Graph Signal Processing: Filterbanks, Sampling and Applications to Machine Learning and Video Coding

Antonio Ortega

Signal and Image Processing Institute
Department of Electrical Engineering
University of Southern California
Los Angeles, California

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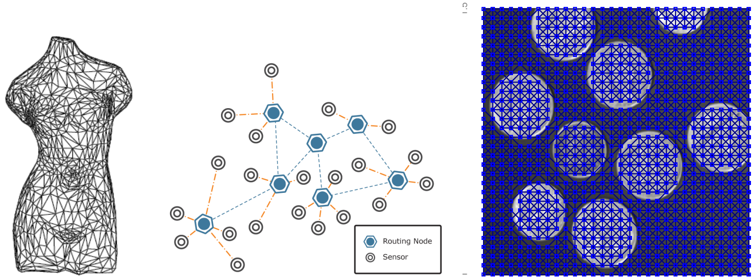
Outline

Motivation

Graphs provide a flexible model to represent many datasets:

- Examples in Euclidean domains

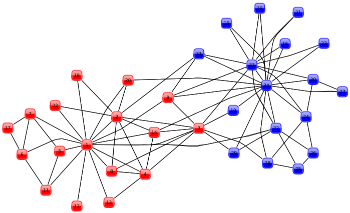


(a) (b) (c)

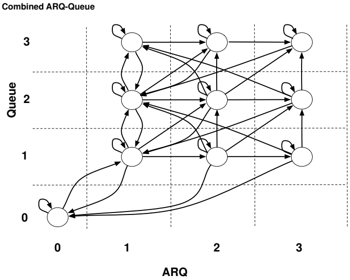
(a) Computer graphics[1] (b) Wireless sensor networks [2] (c) image - graphs

Motivation

- Examples in non-Euclidean settings



(a)



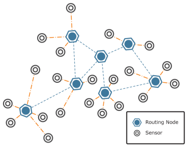
(b)

(a) Social Networks [3], (b) Finite State Machines(FSM)

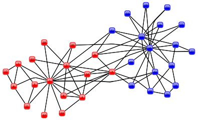
Graph Signal Processing

- Given a graph (fixed or learned from data)
- and given signals on the graph (set of scalars associated to vertices)
- define frequency, sampling, transforms, etc
- in order to solve problems such as compression, denoising, interpolation, etc
- Overview papers:
  [Shuman, Narang, Frossard, Ortega, Vandergheysnt, SPM'2013]
  [Sandryhaila and Moura 2013]

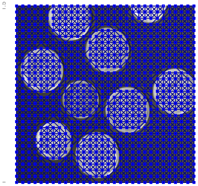
Examples



- Sensor network
  - Relative positions of sensors (kNN), temperature
  - Does temperature vary smoothly?



- Social network
  - Friendship relationship, age
  - Are friends of similar age?



- Images
  - Pixel positions and similarity, pixel values
  - Discontinuities and smoothness

Sampling

- Sample signal at discrete points in time/space
- Core tool in Digital Signal Processing (DSP)
  - From Analog to Digital
  - Digital to Digital
- Many pervasive applications
  - Digital audio (CDs, MP3s, etc)
  - Digital images/video (JPEG, MPEG, AVC, HEVC)

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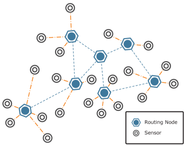
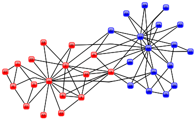
- Many pervasive applications
  - Digital audio (CDs, MP3s, etc)
  - Digital images/video (JPEG, MPEG, AVC, HEVC)
- Key concept: signals can be recovered from their samples
- **Questions**:
  - **What properties enable recovery?**
  - **How to sample?**
  - **How to reconstruct?**

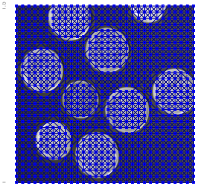
Graph Signal Sampling Examples



- Sensor network
  - Relative positions of sensors (kNN), temperature
  - Measure temperature in a subset of sensors



- Social network
  - Friendship relationship, age
  - Estimate interests for a subset of users

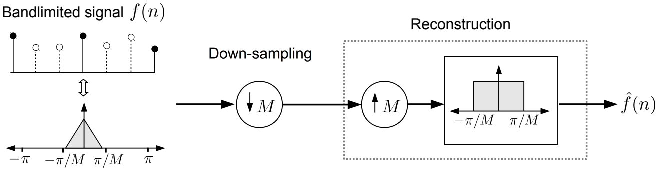


- Images
  - Pixel positions and similarity, pixel values
  - New image sampling techniques

Sampling

Traditional DSP

- Samples dropped in a regular fashion, spectral folding (aliasing).
- Cutoff frequency $\Leftrightarrow$ sampling rate.



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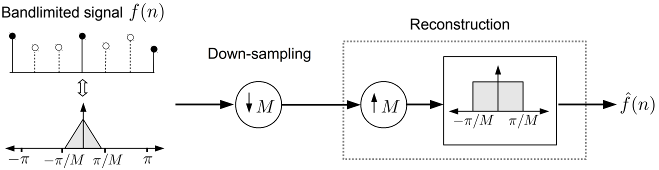
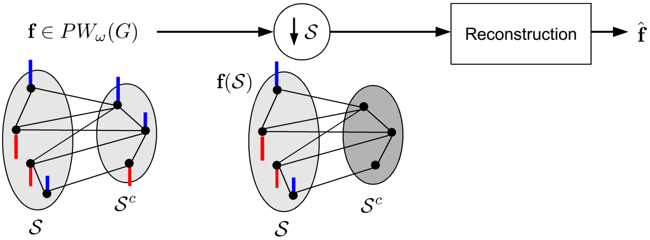


Answers:

- What properties enable recovery? Signals are smooth, low frequency
- How to sample? Regular sampling
- How to reconstruct? Low pass filtering

Graph Sampling?

- Measure a few nodes to estimate information throughout the graph
- Reconstruct signal in whole graph



Graph Sampling?

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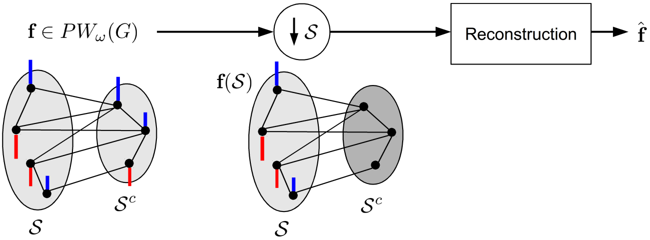
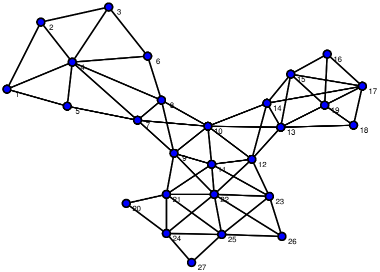


Questions:

- What properties enable recovery? Need to define frequency
- How to sample? No obvious regular sampling
- How to reconstruct? Filtering is needed

Outline

Graphs 101



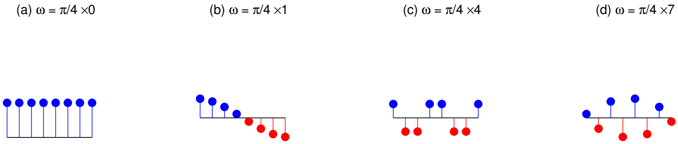
- Graph $G = (\mathcal{V}, E, w)$.
- Adjacency $\mathbf{A}$, $a_{ij} = a_{ji} =$ weight of link between i and j.
- Degree $\mathbf{D} = diag\{d_i\}$
- Laplacian matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$.
- Symmetric normalized Laplacian $\mathcal{L} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$
- Graph Signal $\mathbf{f} = \{f(1), f(2), ..., f(N)\}$

- Assumptions:
  1. Undirected graphs without self loops.
  2. Scalar sample values

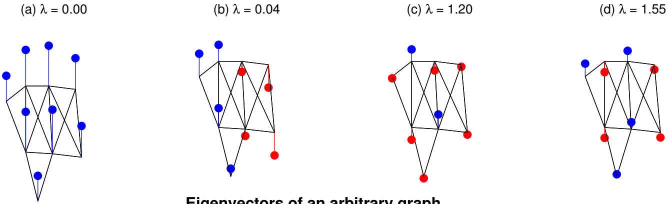
Spectrum of Graphs

- Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}'$
- Eigenvectors of $\mathbf{L}$: $\mathbf{U} = \{\mathbf{u}_k\}_{k=1:N}$
- Eigenvalues of $\mathbf{L}$: $diag\{\boldsymbol{\Lambda}\} = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$
- **Eigen-pair system $\{(\lambda_k, \mathbf{u}_k)\}$ provides Fourier-like interpretation — Graph Fourier Transform (GFT)**

Graph Frequencies

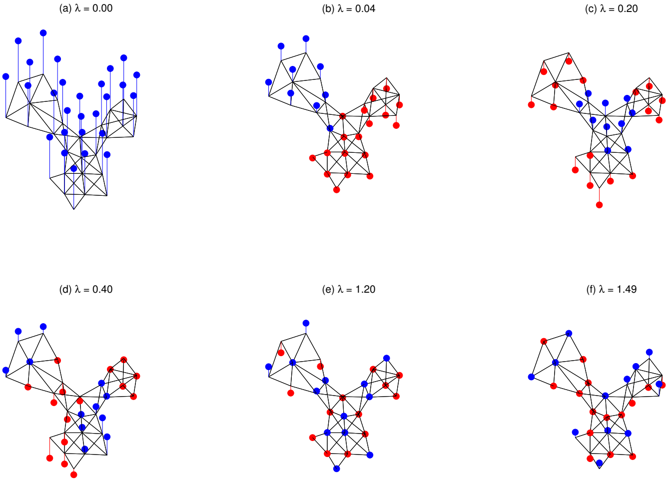


DCT basis for regular signals

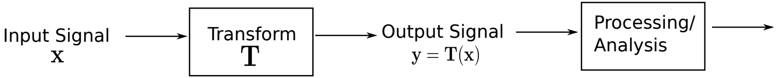


Eigenvectors of an arbitrary graph

Eigenvectors of graph Laplacian



Graph Transforms and Filters



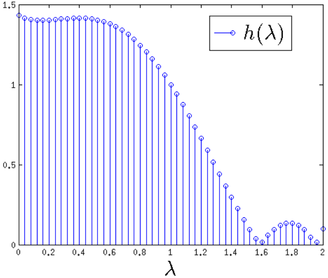
- Desirable properties
  - Invertible
  - Critically sampled
  - Orthogonal

- **What makes these “graph transforms”?**
  - Frequency interpretation
  - Vertex localization

Frequency interpretation: SGWT

- Spectral Wavelet transforms [Hammond et al. 2011]:

  Design spectral kernels: $h(\lambda) : \sigma(G) \rightarrow \mathbb{R}$.



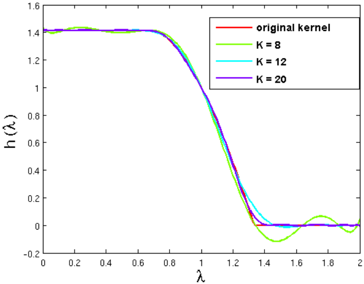
$$\mathbf{T}_h = h(\mathcal{L}) = \mathbf{U}h(\mathbf{\Lambda})\mathbf{U}^t$$

$$h(\mathbf{\Lambda}) = diag\{h(\lambda_i)\}$$

- Analogy: FFT implementation of filters

Vertex Localization: SGWT

- Polynomial kernel approximation:



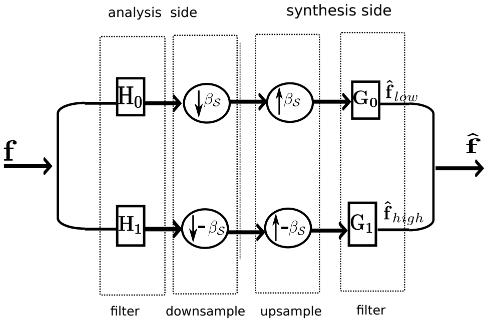
$$h(\lambda) \approx \sum_{k=0}^{K} a_k \lambda^k$$

$$\mathbf{T}_h \approx \sum_{k=0}^{K} a_k \mathcal{L}^k$$

K-hop localized: no spectral decomposition required.

Graph Filterbank Designs

- Formulation of critically sampled graph filterbank design problem
- Design filters using spectral techniques [Hammond et al. 2009].
- Orthogonal (not compactly supported) [Narang and O., IEEE TSP June 2012]
- Bi-Orthogonal (compactly supported) [Narang and O., IEEE TSP Oct 2013]



Bipartite Subgraph Decomposition

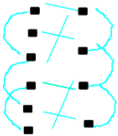
- These designs work on bipartite graphs
- But not all graphs are bipartite...

Bipartite Subgraph Decomposition

- These designs work on bipartite graphs
- But not all graphs are bipartite...
- Solution: "Iteratively" decompose non-bipartite graph G into K bipartite subgraphs

Bipartite Subgraph Decomposition

- Example of a 2-dimensional ($K = 2$) decomposition:



G_0

Bipartite Subgraph Decomposition

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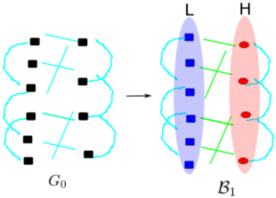

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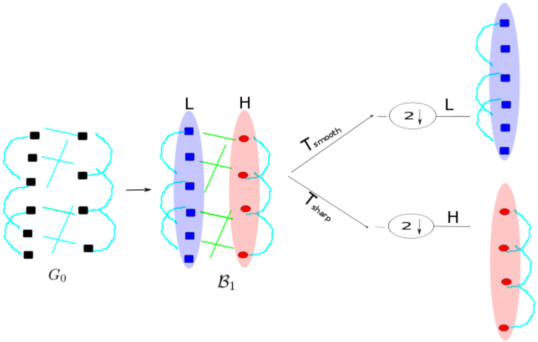

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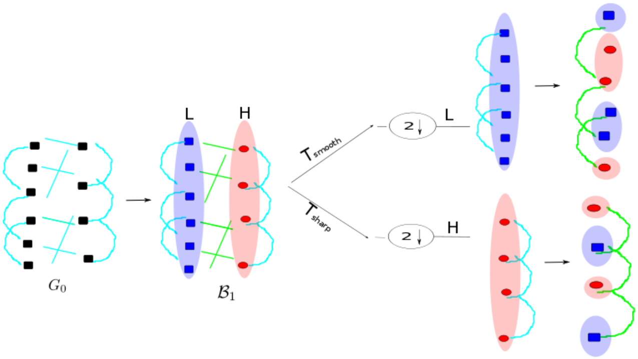

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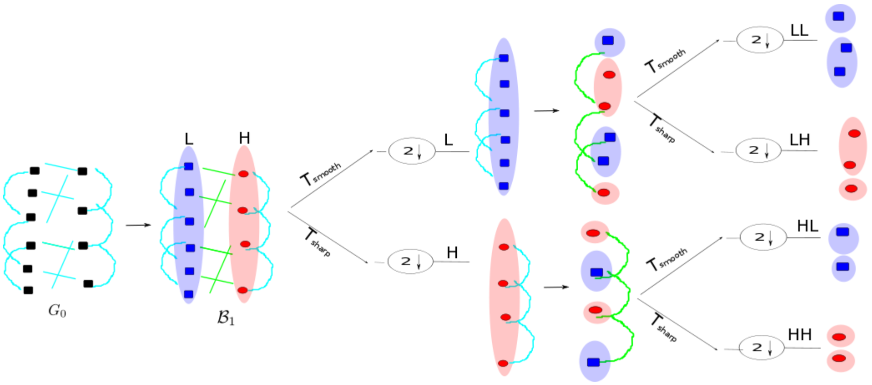
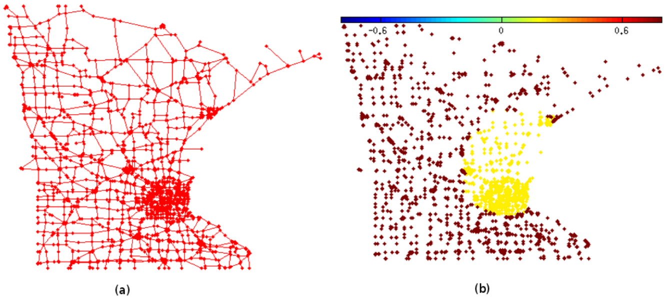
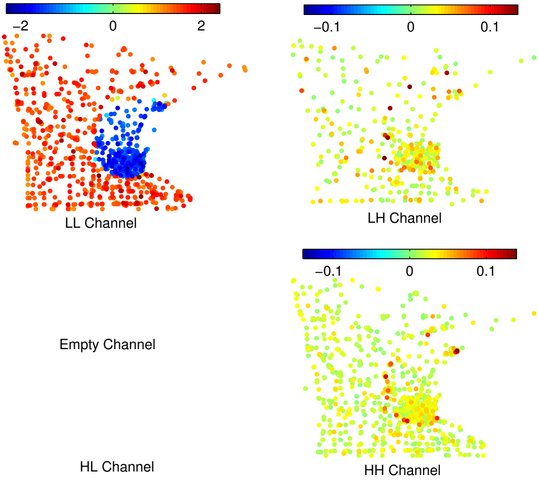


Example



Minnesota traffic graph and graph signal

Example

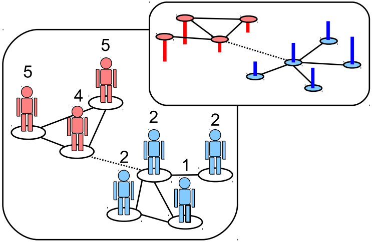


Output coefficients of the proposed filterbanks with parameter $m = 24$.

Outline

Bandlimited signals

- In many applications, signals of interest are *smooth*.
- Smooth signals $\rightarrow$ lowpass in spectral domain.



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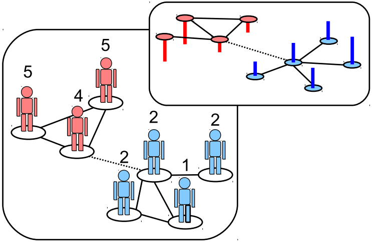
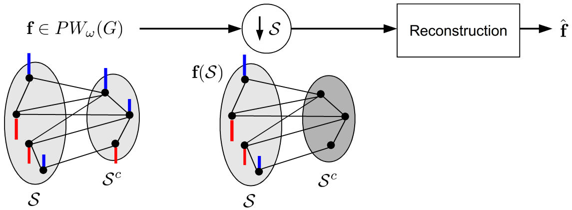


Bandlimited signals in graphs

- ω*-bandlimited signal*: GFT has support $[0, \omega]$.
- *Paley-Wiener space* $PW_\omega(G)$: Space of all ω-bandlimited signals.
  - $PW_\omega(G)$ is a subspace of $\mathbb{R}^N$.
  - $\omega_1 \leq \omega_2 \Rightarrow PW_{\omega_1}(G) \subseteq PW_{\omega_2}(G)$.

Sampling graph signals

- Input signal: $\mathbf{f} \in PW_\omega(G)$.
- Sampling set: $\mathcal{S} \subset \mathcal{V}$, unknown set: $\mathcal{S}^c$.
- Sampled signal: $\mathbf{f}(\mathcal{S}) \in \mathbb{R}^{|\mathcal{S}|}$.



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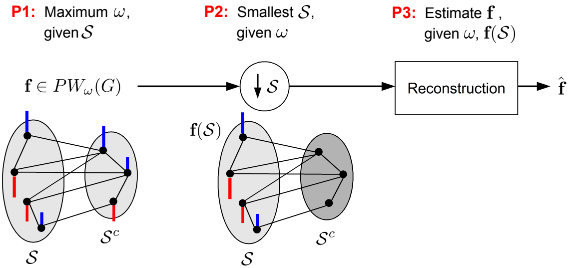
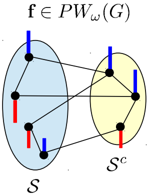


P1: Given $\mathcal{S}$, *maximum* ω?

P2: Given ω, *smallest set* $\mathcal{S}$?

P3: Given ω and $\mathbf{f}(\mathcal{S})$, how to recover $\mathbf{f}$?

Necessary and sufficient condition [Anis, Gadde, O., ICASSP '14]



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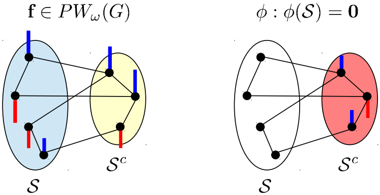

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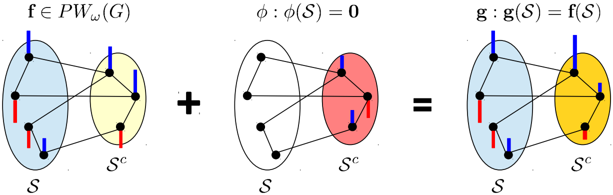

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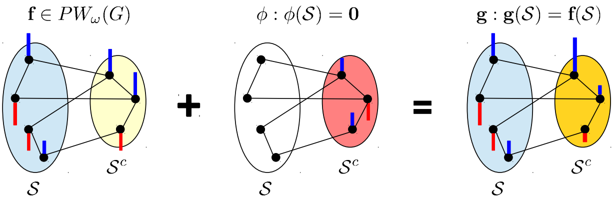

- If $\phi \in PW_\omega(G)$, then $\mathbf{g} = \mathbf{f} + \phi \in PW_\omega(G)$.
- $\mathbf{f} \neq \mathbf{g}$ and $\mathbf{f}(\mathcal{S}) = \mathbf{g}(\mathcal{S}) \Rightarrow$ *trouble!*

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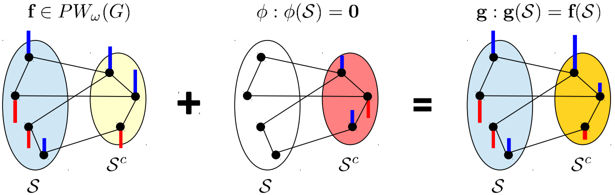

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Lemma

Let $L_2(\mathcal{S}^c) = \{\phi : \phi(\mathcal{S}) = \mathbf{0}\}$. All signals $\mathbf{f} \in PW_\omega(G)$ can be perfectly recovered from $\mathcal{S}$ if and only if $PW_\omega(G) \cap L_2(\mathcal{S}^c) = \{\mathbf{0}\}$.

Sampling theorem [Anis, Gadde, O., ICASSP '14]

Theorem (Sampling theorem)

All signals $\mathbf{f} \in PW_\omega(G)$ can be perfectly recovered from their samples $\mathbf{f}(\mathcal{S})$ if and only if

$$\omega < \inf_{\phi \in L_2(\mathcal{S}^c)} \omega(\phi) \triangleq \omega_c(\mathcal{S})$$

We call $\omega_c(\mathcal{S})$ the true cutoff frequency.

The cutoff frequency depends on the size of $\mathcal{S}$ and topologies of G and $\mathcal{S}$.

In order to optimize sampling set, maximize cut-off frequency

How to optimize the sampling set?

Cutoff frequency $\equiv$ bandwidth of smoothest signal ϕ^* in $L_2(\mathcal{S}^c)$.

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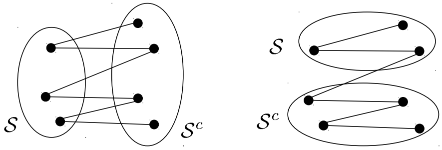

Which choice of $\mathcal{S}$ leads to a higher cutoff frequency?

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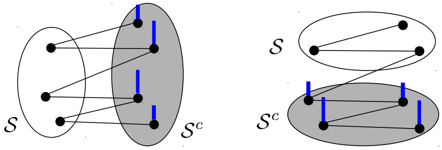

Which choice of $\mathcal{S}$ leads to a higher cutoff frequency?

- Compare $\phi^* \in L_2(\mathcal{S}^c)$ for both graphs.
- More cross-links $\Rightarrow$ higher variation $\Rightarrow$ higher bandwidth.

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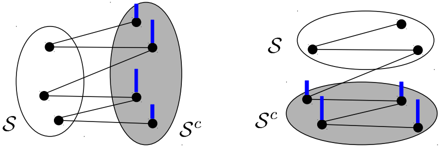
Cutoff frequency $\equiv$ bandwidth of smoothest signal ϕ^* in $L_2(\mathcal{S}^c)$.



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P2: Smallest sampling set

Formulation

Relax the true cutoff $\omega_c(\mathcal{S})$ by $\Omega_k(\mathcal{S})$, an approximation based on the k-th power of the Laplacian and solve:

$$\text{Minimize } \ |\mathcal{S}| \ \text{ subject to } \quad \Omega_k(\mathcal{S}) \geq \omega_c$$

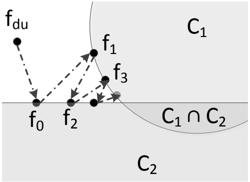
Greedy Approach to get an estimate of $\mathcal{S}_{\text{opt}}$ [Anis, Gadde and O., ICASSP 2014]:

- Start with $\mathcal{S} = \{\emptyset\}$.
- Add nodes to $\mathcal{S}$ (from $\mathcal{S}^c$) one-by-one that ensure maximum increase in $\Omega_k(\mathcal{S})$ at each step.
- Essentially this involves finding nodes that are "far" from $\mathcal{S}$ at each iteration

- Note: Most alternative proposed methods require knowledge of the GFT

P3: Signal Reconstruction

- $\mathcal{C}_1 = \{\mathbf{x} : \mathbf{x}(\mathcal{S}) = \mathbf{f}(\mathcal{S})\}$ and $\mathcal{C}_2 = PW_\omega(G)$.
- We need to find a unique $\mathbf{f} \in \mathcal{C}_1 \cap \mathcal{C}_2 \Rightarrow$ sampling theorem guarantees uniqueness.



Projection onto convex sets

$\mathbf{f}_{i+1} = \mathbf{P}_{\mathcal{C}_2}\mathbf{P}_{\mathcal{C}_1}\mathbf{f}_i$, where $\mathbf{f}_0 = [\mathbf{f}(\mathcal{S})^\top, \mathbf{0}]^\top$.

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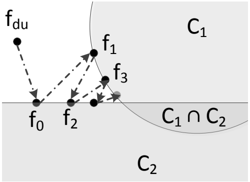
- $\mathcal{C}_1 = \{\mathbf{x} : \mathbf{x}(\mathcal{S}) = \mathbf{f}(\mathcal{S})\}$ and $\mathcal{C}_2 = PW_\omega(G)$.
- We need to find a unique $\mathbf{f} \in \mathcal{C}_1 \cap \mathcal{C}_2 \Rightarrow$ sampling theorem guarantees uniqueness.



Projection onto convex sets

$\mathbf{f}_{i+1} = \mathbf{P}_{\mathcal{C}_2}\mathbf{P}_{\mathcal{C}_1}\mathbf{f}_i$, where $\mathbf{f}_0 = [\mathbf{f}(\mathcal{S})^\top, \mathbf{0}]^\top$.

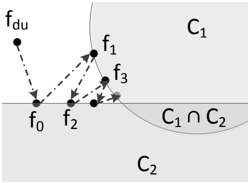
- $\mathbf{P}_{\mathcal{C}_1}$ resets the samples on $\mathcal{S}$ to $\mathbf{f}(\mathcal{S})$.

- $\mathbf{P}_{\mathcal{C}_2} = \mathbf{U}h(\mathbf{\Lambda})\mathbf{U}^\top$ sets $\tilde{\mathbf{f}}(\lambda) = 0$ if $\lambda > \omega$.

$$h(\lambda) = \begin{cases} 1, & \text{if } \lambda < \omega \\ 0, & \text{if } \lambda \geq \omega \end{cases}$$

P3: Signal Reconstruction

- $\mathcal{C}_1 = \{\mathbf{x} : \mathbf{x}(\mathcal{S}) = \mathbf{f}(\mathcal{S})\}$ and $\mathcal{C}_2 = PW_\omega(G)$.
- We need to find a unique $\mathbf{f} \in \mathcal{C}_1 \cap \mathcal{C}_2 \Rightarrow$ sampling theorem guarantees uniqueness.



Projection onto convex sets

$\mathbf{f}_{i+1} = \mathbf{P}_{\mathcal{C}_2}\mathbf{P}_{\mathcal{C}_1}\mathbf{f}_i$, where $\mathbf{f}_0 = [\mathbf{f}(\mathcal{S})^\top, \mathbf{0}]^\top$.

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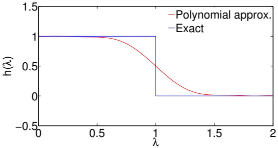


- $\mathbf{P}_{\mathcal{C}_2} \approx \sum_{i=1}^n \left(\sum_{j=0}^p a_j \lambda_i^j\right) \mathbf{u}_i \mathbf{u}_i^\top = \sum_{j=0}^p a_j \boldsymbol{\mathcal{L}}^j \rightarrow p$-hop localized

Outline

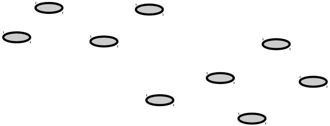
Learning: Motivation and Problem Definition

- Unlabeled data is abundant. Labeled data is expensive and scarce.
- Solution: **Active Semi-supervised Learning** (SSL).
- **Problem setting**: Offline, pool-based, batch-mode active SSL via graphs

- How to predict unknown labels from the known labels?
- What is the optimal set of nodes to label given the learning algorithm?

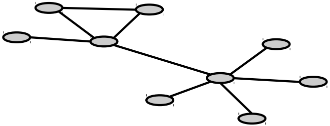
Graph-based semi-supervised learning

Key Idea:



Graph-based semi-supervised learning

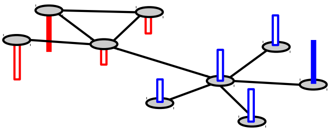
Key Idea:



- Construct a distance-based similarity graph.
  - Data points $\rightarrow$ nodes.
  - Edge weights $\rightarrow$ similarity.

Graph-based semi-supervised learning

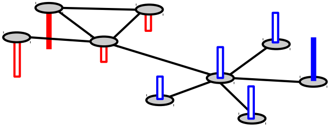
Key Idea:



- Construct a distance-based similarity graph.
  - Data points $\rightarrow$ nodes.
  - Edge weights $\rightarrow$ similarity.
- Treat class indicator vectors as *graph signals* that are
  - Consistent with known labels.
  - Smooth on the graph.

Graph-based semi-supervised learning

Key Idea:



- Construct a distance-based similarity graph.
  - Data points $\rightarrow$ nodes.
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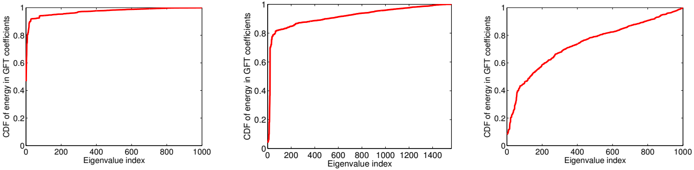
  - Smooth on the graph.

Why does this work?

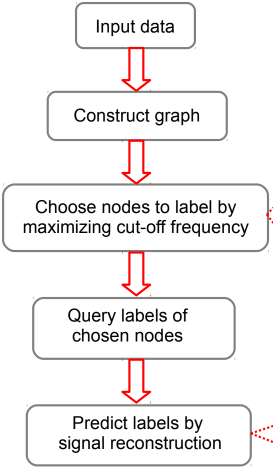
Connection to Graph Sampling

- *Class membership functions can be approximated by bandlimited graph signals.*



(a) USPS (b) Isolet (c) 20 newsgroups

Summary of the Algorithm [Anis, Gadde, O., KDD 2014]



Input: $G = \{\mathcal{V}, E\}, \mathbf{L}$, target size m, parameter $k \in \mathbb{Z}^+$

Initialize: $\mathcal{S} = \{\emptyset\}$

while $|\mathcal{S}| \leq m$

For $\mathcal{S}$, compute the smoothest signal $\phi_k^{\text{opt}} \in L_2(\mathcal{S}^c)$

$v \leftarrow \arg\max_i \left[(\phi_k^{\text{opt}}(i))^2\right]$

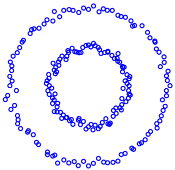
$\mathcal{S} \leftarrow \mathcal{S} \cup v$

end while

POCS iteration: $\mathbf{f}_{i+1} = \mathbf{P}_{\mathcal{C}_1}\mathbf{P}_{\mathcal{C}_2}\mathbf{f}_i$

Label of node $n = \arg\max_c \mathbf{f}^c(n)$

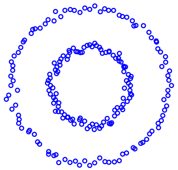
Results: Toy Example



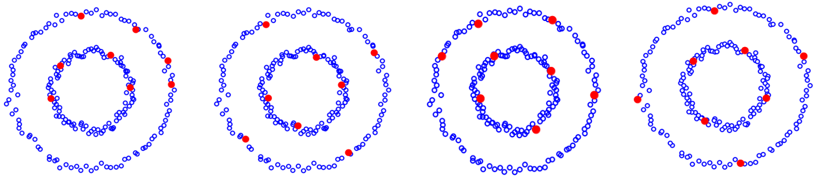
Task

Pick 8 data points for labeling.

Results: Toy Example

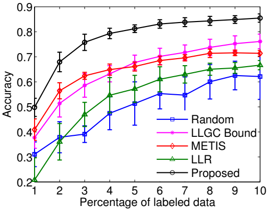


Task
Pick 8 data points for labeling.

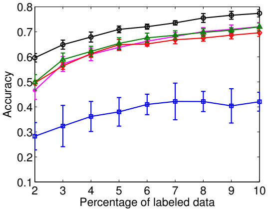


- 4 data points picked from each circle.
- Maximally separated points within one circle.
- Maximal spacing between selected data points in different circles.

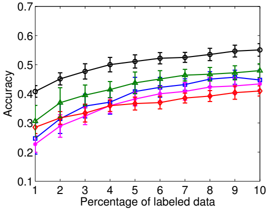
Results: Real Datasets



- USPS: handwritten digits
- $\mathbf{x}_i = 16 \times 16$ image
- number of classes $= 10$
- K-NN graph with $K = 10$
- $w_{ij} = \exp\left(-\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2}{2\sigma^2}\right)$



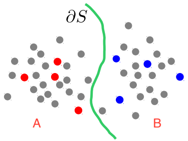
- ISOLET: spoken letters
- $\mathbf{x}_i \in \mathbb{R}^{617}$ speech features.
- number of classes $= 26$
- K-NN graph with $K = 10$
- $w_{ij} = \exp\left(-\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2}{2\sigma^2}\right)$



- Newsgroups: documents
- $\mathbf{x}_i \in \mathbb{R}^{3000}$ tf-idf of words
- number of classes $= 10$
- K-NN graph with $K = 10$
- $w_{ij} = \frac{\mathbf{x}_i^\top \mathbf{x}_j}{\|\mathbf{x}_i\| \|\mathbf{x}_j\|}$

Why bandlimited signals?

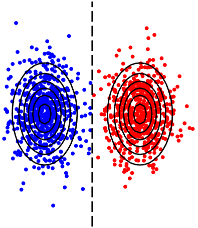
- Data model:
  - Set of points $X = \{\mathbf{X}_1, \ldots, \mathbf{X}_n\}, \mathbf{X}_i \in \mathbb{R}^d$ drawn *i.i.d.* from $p(\mathbf{x})$.
  - Smooth hypersurface ∂S dividing $\mathbb{R}^d$ into two parts: A and B.
  - Indicator vector for points in A: $\mathbf{1}_A \in \{0,1\}^n$ such that $\mathbf{1}_A(i) = 1$ if $\mathbf{X}_i \in A$ and 0 otherwise.

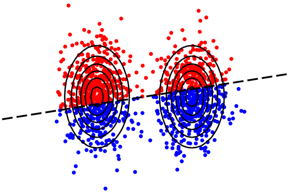


- Asymptotic Result [Anis et al., ICASSP 2015]:
  - Bandwidth $\omega(\mathbf{1}_S)$ function of $\max(p(\mathbf{x}))$ along δS

Implications: interpretation of bandwidth

- If boundary passes through regions of low density:
  - Bandwidth of indicator is low
  - Sampling theory $\Rightarrow$ fewer known labels required for perfect reconstruction.





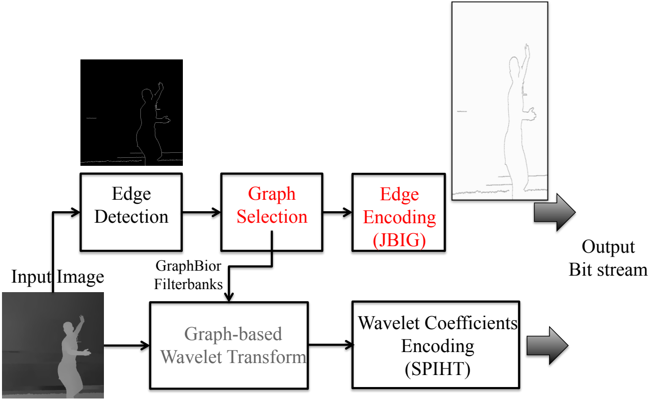
Outline

Applying Graph-Based Methods to Image/Video Coding

- Main idea
  - Images can be viewed as regular (4 connected, 8 connected) pixel graphs
  - Making graphs irregular (different weights) allows us to capture different levels of correlation between pixels
- Two main ideas:
  - Use side information to signal discontinuities
  - Find "average" graphs as alternatives to KLT

Depth Image Coding [Narang, Chao and O., APSIPA 2013]

- Block Diagram

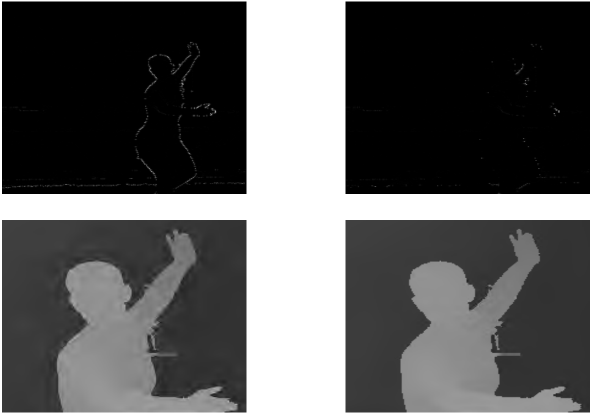


- Link-weights can be adjusted to reflect geometrical structure of the image.

Depth Image Coding [Narang, Chao and O., APSIPA 2013]

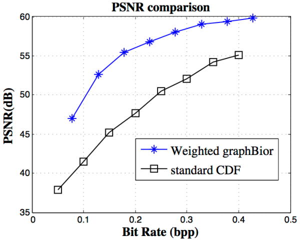
CDF 9/7

Graph 9/7



Depth Image Coding [Narang, Chao and O., APSIPA 2013]

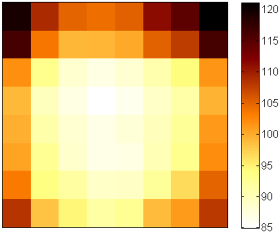
- Edge detection: Prewitt
- Laplacian Normalization: Random Walk Laplacian
- Filterbanks: GraphBior 4/3 and CDF 9/7
- Unreliable Link Weight: 0.01
- Transform level: 5
- Encoder: SPIHT



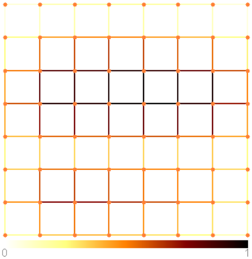
Residual Characteristics of Inter-predicted residuals

Main observation:

- Residual samples around block boundaries have higher energy
- Occlusions, partial mismatches between reference and predicted blocks



Variance



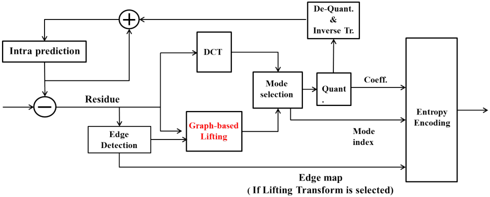
Graph

- Developed tools to learn sparse graphs from data
- Can outperform KLT (robustness)

Lifting Approximation for Intra-predicted videos

(Chao et al, submitted to ICASSP 2016)

- Application on residual video frames extracted from HEVC standard
- Block size: 8×8
- Hybrid mode between Lifting and DCT: based on RD cost
  $SSE + \lambda \times Bitrate$
- Test Set $\{Foreman, Mobile, Silent, Deadline\}$



Lifting Approximation for Intra-predicted videos

(Chao et al, submitted to ICASSP 2016)

- Lifting/DCT hybrid with graph sampling approximates GFT/DCT hybrid approach
- Sampling (requires eigen-decomposition) can be well approximated with heuristic approach, e.g. greedy MaxCut.
- Online transform complexity against GFT: $O(N \log N)$ v.s. $O(N^2)$

- Bjontegaard Distortion-Rate against DCT

Methods	GFT		Lifting with GMRF sampling		Lifting (Max Cut w/ re-connection)	
	Δ PSNR (dB)	**Δ rate (%)**	**Δ PSNR (dB)**	**Δ rate (%)**	**Δ PSNR (dB)**	**Δ rate (%)**
Foreman	0.34	-7.28	0.29	-6.42	0.26	-5.77
Mobile	0.17	-1.46	0.10	-0.97	0.10	-0.96
Silent	0.22	-4.28	0.20	-3.88	0.18	-3.58
Deadline	0.37	-4.97	0.31	-3.98	0.30	-3.90

Outline

Introduction

Basic Concepts

Sampling

Application: Learning

Application: Image/Video Processing

Conclusions

Conclusions

- Extending signal processing methods to arbitrary graphs: Downsampling, Space-frequency, Multiresolution, Wavelets
- Many open questions: very diverse types of graphs, results may apply to special classes only
- Outcomes
  - Work with massive graph-datasets: potential benefits of localized "frequency" analysis
  - Novel insights about traditional applications (image/video processing)
  - Promising results in machine learning, image processing
- To get started:
  [Shuman, Narang, Frossard, Ortega, Vandergheysnt, SPM'2013]
  [Sandryhaila and Moura 2013]

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