# Integer Multichannel Transform 

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#### Abstract

In this paper, we develop an algorithm that can convert any reversible multichannel system into a reversible integer multichannel transform. The integer transform means the operation whose inputs and outputs are all sums of powers of two. Recently, the triangular matrix scheme was shown to be able to convert any nonsingular discrete transform into a reversible discrete integer transform. Since as other discrete transform, the multichannel system can also be expressed as a matrix form, we suggest that the triangular matrix scheme can also be applied for converting a multichannel system into a reversible integer transform. The proposed methods are useful for multirate signal processing, wavelet analysis, communication, and image processing.


## I. INTRODUCTION

The reversible integer transform [1-8] is an operator that satisfies the following constraints:
(Constraint 1) Suppose that $x[n]$ and $y[n]$ are the input and the output of the integer transform, respectively. If $x[n]$ is a sum of powers of 2 (SOPOT):

$$
\begin{equation*}
x[n]= \pm \sum_{k} c_{n, k} 2^{-k} \quad \text { where } c_{n, k}=0 \text { or } 1, \tag{1}
\end{equation*}
$$

then $y[n]$ should also be a SOPOT:

$$
\begin{equation*}
y[n]= \pm \sum_{k} b_{n, k} 2^{-k} \quad \text { where } b_{n, k}=0 \text { or } 1 . \tag{2}
\end{equation*}
$$

(Constraint 2) The inverse integer transform also has the integer-to-integer property. That is, if $y_{1}[n]$ and $x_{1}[n]$ are the input and output of the inverse integer transform, when $y_{1}[n]$ is a SOPOT, then $x_{1}[n]$ is also a SOPOT.
(Constraint 3) If $x[n]$ and $y[n]$ are the input and the output of the forward integer transform, one can perfectly reconstruct the input $x[n]$ from $y[n]$ by the inverse integer transform without any loss.
For example, the Walsh (Hadamard) transform, the Haar transform, and the Jacket transform [3] are integer transforms. The reversible integer transform is easier to implement by the fixed-point processor and more suitable for VLSI design. However, most operations, such as the discrete Fourier transform, the discrete wavelet transform, and the discrete cosine transform, are non-integer transforms. It is an interesting problem that how to convert a non-integer transform into an integer transform without the loss of perfect reconstruction property.

In [1], based on the dyadic symmetry property, the integer cosine transform was derived. Then, in [4], the integer RGB-to- YCbCr transform was proposed and it is widely used in the

JPEG2000 standard and other advanced color image compression algorithms.

In [2], Hao and Shi used the lifting scheme [5] and triangular matrix decomposition to convert any discrete linear operation into the reversible integer transform. If a discrete operation can be expressed as the following matrix form

$$
\begin{equation*}
\mathbf{y}=\mathbf{A} \mathbf{x} \quad \text { where } \operatorname{det}(\mathbf{A}) \neq 0 \tag{3}
\end{equation*}
$$

then, using the algorithm in [2], it can be converted into a reversible integer transform. Based on the lifting scheme, the integer DFT [6], the integer KLT [7], and the integer color transform [8] were derived successfully.

Until now, the development of the reversible integer transform focuses on the discrete operation that has the matrix form in (3). In this paper, we extend the concept of the reversible integer transform and discuss how to convert a multichannel filter system into a reversible integer transform. A multichannel filter system can be expressed as

$$
\begin{align*}
y_{p}[n]= & h_{p, 1}[n] * x_{1}[n]+h_{p, 2}[n] * x_{2}[n]+h_{p, 3}[n] * x_{3}[n]+ \\
& \cdots \cdots+h_{p, P}[n] * x_{P}[n] \tag{4}
\end{align*}
$$

where $p=1,2, \ldots, P, *$ means the convolution, and $h_{p, q}[n]$ can be an infinite impulse response (IIR) or a finite impulse response (FIR) filter. With the $Z$ transform, the multichannel filter system can be rewritten as

$$
\left[\begin{array}{c}
Y_{1}(z)  \tag{5}\\
Y_{2}(z) \\
Y_{3}(z) \\
\vdots \\
Y_{P}(z)
\end{array}\right]=\left[\begin{array}{ccccc}
H_{1,1}(z) & H_{1,2}(z) & H_{1,3}(z) & \cdots & H_{1, P}(z) \\
H_{2,1}(z) & H_{2,2}(z) & H_{2,3}(z) & \cdots & H_{2, P}(z) \\
H_{3,1}(z) & H_{3,2}(z) & H_{3,3}(z) & \cdots & H_{3, P}(z) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
H_{P, 1}(z) & H_{P, 2}(z) & H_{P, 3}(z) & \cdots & H_{P, P}(z)
\end{array}\right]\left[\begin{array}{c}
X_{1}(z) \\
X_{2}(z) \\
X_{3}(z) \\
\vdots \\
X_{P}(z)
\end{array}\right] .
$$

It is similar to the matrix operation in (3), but each entry is the $Z$-transform of a filter. In this paper, we propose a way to convert (5) into a reversible integer transform. With the proposed method, one can convert any multichannel system into a reversible integer multichannel transform successfully if the determinant of leading coefficients is nonzero.

## II. Reversible Integer Filter in One Channel Case

Before discussing the multichannel case, we first discuss how to convert a one-channel IIR / FIR filter into a reversible integer transform. Any IIR filter can be expressed as:

$$
\begin{align*}
y[n] & =a_{0} x[n+\tau]+a_{1} x[n+\tau-1]+\cdots \cdots+a_{N} x[n+\tau-N] \\
& -d_{1} y[n-1]-d_{2} y[n-2]-\cdots \cdots-d_{M} y[n-M] . \tag{6}
\end{align*}
$$

Most IIR filters are not reversible integer transforms, since in usual $a_{n}$ and $d_{m}$ are not sums of powers of 2 . Although one can perform truncation for $a_{n}$ and $d_{m}$ to make them sums of powers of 2 , if $a_{0} \neq \pm 2^{k}$, Constraint 2 is still not satisfied. Note that, to recover $x[n]$ from $y[n]$, we should perform

$$
\begin{align*}
& x[n]=a_{0}^{-1} y[n-\tau]+a_{0}^{-1} d_{1} y[n-\tau-1]+\cdots \cdots+a_{0}^{-1} d_{M} y[n-\tau-M] \\
& \quad-a_{0}^{-1} a_{1} x[n-1]-a_{0}^{-1} a_{2} x[n-2]-\cdots \cdots-a_{0}^{-1} a_{N} x[n-N] . \tag{7}
\end{align*}
$$

For example, if $a_{0}=5 / 2$, then $a_{0}^{-1}=2 / 5$, which cannot be expressed as a sum of powers of 2 and Constraint 2 is not satisfied.

In the following, we introduce a method to convert an IIR filter into a reversible integer transform. Note that, since the FIR filter is a special case of the IIR filter where $d_{1}=d_{2}=\ldots$ $=d_{M}=0$ in (6) and (7), the following discussions can also be applied to the FIR filter case.

First, we scale the coefficients of $x[n+\tau-k]$ in (6) as:

$$
\begin{gather*}
y_{a}[n]=\sigma a_{0} x[n+\tau]+\sigma a_{1} x[n+\tau-1]+\cdots \cdots+\sigma a_{N} x[n+\tau-N] \\
-d_{1} y_{a}[n-1]-d_{2} y_{a}[n-2]-\cdots \cdots-d_{M} y_{a}[n-M]  \tag{8}\\
\text { where } \sigma=\left|2^{k} / a_{0}\right|, k \text { is some integer. }
\end{gather*}
$$

Then we perform truncation operations for $\sigma a_{n}$ and $d_{m}$

$$
\begin{equation*}
s_{n}=Q_{b_{1}}\left(\sigma a_{n}\right), \quad c_{m}=Q_{b_{1}}\left(d_{m}\right) \tag{9}
\end{equation*}
$$

where $Q()$ is the truncation operation:

$$
\begin{equation*}
Q_{b_{1}}\left(v_{n}\right)=2^{-b_{1}} \operatorname{round}\left(2^{b_{1}} v_{n}\right) \tag{10}
\end{equation*}
$$

After truncation, both $s_{n}$ and $c_{m}$ are sums of powers of 2 . Then, we can rewrite the forward and inverse IIR filters in (6) and (7) as the following reversible integer transform pair:

$$
\begin{align*}
t[n]= & s 2^{k} x[n+\tau]+s_{1} x[n+\tau-1]+\cdots \cdots+s_{N} x[n+\tau-N] \\
& -c_{1} t[n-1]-c_{2} t[n-2]-\cdots \cdots-c_{M} t[n-M]  \tag{11}\\
x[n]= & s 2^{-k}\left\{t[n-\tau]+c_{1} t[n-\tau-1]+\cdots \cdots+c_{M} t[n-\tau-M]\right. \\
& \left.-s_{1} x[n-1]-s_{2} x[n-2]-\cdots \cdots-s_{N} x[n-N]\right\} \tag{12}
\end{align*}
$$

where $s=1$ if $a_{0}>0$ and $s=-1$ if $a_{0}<0$.
Note that, in (11),

$$
\begin{equation*}
t[n] \approx y_{a}[n]=\sigma y[n] . \tag{13}
\end{equation*}
$$

Since $t[n]$ is near to $y[n]$ multiplied by a constant, the performance of the IIR filter in (11) is still similar to that of the original IIR filter. Moreover, it is no hard to show that (11) and (12) satisfy all of the three constraints in Section 1. Therefore, they form a reversible integer IIR filter pair.

## III. Reversible Integer Multichannel Transforms in One-Nontrivial Row Cases

For the multichannel filter case in (5), the problem of reversible integer transform conversion becomes more complicated. However, we can still use the method analogous to the triangular matrix decomposition in [2] to convert it into the reversible integer multichannel transform. Before discussing the general case, we first discuss a special case: the one-nontrivial row multichannel system. It has the form of
$\mathbf{E}_{\mathbf{p}}=\left[\begin{array}{cccccccc}1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ E_{p, 1}(z) & E_{p, 2}(z) & \cdots & E_{p, p-1}(z) & E_{p, p}(z) & E_{p, p+1}(z) & \cdots & E_{p, N}(z) \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1\end{array}\right]$
[Theorem 1]: For the one-nontrivial row multichannel system as in (14), suppose that $E_{p, q}(z)$ has the IIR form of

$$
\begin{equation*}
E_{p, q}(z)=\frac{a_{p, q, 0}+a_{p, q, 1} z^{-1}+\cdots \cdots+a_{p, q, N_{p, q}} z^{-N_{p, q}}}{1+d_{p, q, 1} z^{-1}+d_{p, q, 1} z^{-2}+\cdots \cdots+d_{p, q, M_{p, q}} z^{-M_{p, q}}} z^{\tau_{p, q}} . \tag{15}
\end{equation*}
$$

If the leading coefficient is a sum of powers of 2 when $p=q$ :

$$
\begin{equation*}
a_{p, p, 0}=s 2^{k}, \quad s=1 \text { or }-1, \tag{16}
\end{equation*}
$$

then we can use the following way to convert (14) into the reversible integer system.
(A) First, we round the coefficients of $E_{p, q}(z)$ directly and convert $E_{p, q}(z)$ into $T_{p, q}(z)$ :

$$
\begin{equation*}
T_{p, q}(z)=\frac{s_{p, q, 0}+s_{p, q, 1} z^{-1}+\cdots \cdots+s_{p, q, N_{p, q}} z^{-N_{p, q}}}{1+c_{p, q, 1} z^{-1}+c_{p, q, 2} z^{-2}+\cdots \cdots+c_{p, q, M_{p, q}} z^{-M_{p, q}}} z^{\tau_{p, q}} \tag{17}
\end{equation*}
$$

where $s_{p, q, n}=Q_{b}\left(a_{p, q, n}\right), \quad c_{p, q, n}=Q_{b}\left(d_{p, q, n}\right)$,
and the quantization operation $Q_{b}$ is defined in (10).
(B) Then, the forward integer multichannel filter system can be formulated as follows:

$$
\begin{align*}
& y_{q}[n]=x_{q}[n] \quad \text { for } q=1,2, \ldots, P, \quad q \neq p,  \tag{18}\\
& y_{p}[n]=w_{1}[n]+w_{2}[n]+\cdots+w_{P}[n], \tag{19}
\end{align*}
$$

where $w_{q}[n]=s_{p, q, 0} x_{q}\left[n+\tau_{p, q}\right]+s_{p, q, 1} x_{q}\left[n+\tau_{p, q}-1\right]+\cdots \cdots$

$$
\begin{align*}
& +s_{p, q, N_{p, q}} x_{q}\left[n+\tau_{p, q}-N_{p, q}\right]-c_{p, q, 1} w_{q}[n-1]-c_{p, q, 2} w_{q}[n-2] \\
& -\cdots-c_{p, q, M_{p, q}} w_{q}\left[n-M_{p, q}\right] . \tag{20}
\end{align*}
$$

(C) To recover $x_{1}[n], x_{2}[n], \ldots, x_{P}[n]$ from $y_{1}[n], y_{2}[n], \ldots$, $y_{P}[n]$, we can perform the inverse transform as follows:
(i) $x_{q}[n]=y_{q}[n]$ for $q=1,2, \ldots, P, q \neq p$,
(ii) Since $x_{q}[n](q=1,2, \ldots, P, q \neq p)$ has been recovered from (21), we can substitute $x_{q}[n]$ into (20) and calculate $w_{q}[n](q=1,2, \ldots, P, q \neq p)$ recursively from the initiation that $w_{q}[n]=0$ for $n<n_{0}$ if $x[n]=0$ for $n<n_{0}$.
(iii) $g_{p}[n]=y_{p}[n]-w_{1}[n]-w_{2}[n]-\cdots-w_{p-1}[n]-w_{p+1}[n]$

$$
\begin{equation*}
-\cdots-w_{P}[n] . \quad\left(\text { Note that } g_{p}[n]=w_{p}[n]\right) \tag{22}
\end{equation*}
$$

(iv) Then, we can use the way the same as that of the onechannel case (see (12)) to recover $x_{p}[n]$ from $g_{p}[n]$ :

$$
\begin{align*}
& x_{p}[n]=s 2^{-k}\left\{g_{p}\left[n-\tau_{p, p}\right]+c_{p, p, 1} g_{p}\left[n-\tau_{p, p}-1\right]+\cdots \cdots\right. \\
& +c_{p, p, M_{p, p}} g_{p}\left[n-\tau_{p, p}-M_{p, p}\right]-s_{p, p, 1} x_{p}[n-1]- \\
& \left.s_{p, p, 2} x_{p}[n-2]-\cdots \cdots-s_{p, p, N_{p, p}} x_{p}\left[n-N_{p, p}\right]\right\} . \tag{23}
\end{align*}
$$

## IV. Reversible Integer Multichannel Transforms in General Cases

From Section III, if a multichannel system has the onenontrivial row form as in (14) and (16) is satisfied, then we can convert it into the reversible integer system very easily. In the general case, we can first decompose a multichannel system into the combination of the one-row systems with the form of (14) and (16) and use Theorem 1 to convert each part into a reversible integer system. That is, we try to decompose the multichannel system $\mathbf{H}$ in (5) as

$$
\begin{equation*}
\mathbf{H}=\sigma \mathbf{E}_{\mathbf{P}} \mathbf{E}_{\mathbf{P}-\mathbf{1}} \mathbf{E}_{\mathbf{P}-2} \cdots \mathbf{E}_{\mathbf{1}} \mathbf{E}_{\mathbf{0}} \tag{24}
\end{equation*}
$$

where $\mathbf{E}_{1}, \mathbf{E}_{\mathbf{2}}, \ldots$, and $\mathbf{E}_{\mathbf{P}}$ have the one-nontrivial-row form as in (17). That is, $\mathbf{E}_{\mathbf{p}}$ is almost the same as the identity matrix, in addition to the $p^{\text {th }}$ row. The exception is that the nontrivial row of the rightmost matrix $\mathbf{E}_{\mathbf{0}}$ is the last row

$$
\mathbf{E}_{0}=\left[\begin{array}{cccccc}
1 & 0 & 0 & \cdots & 0 & 0  \tag{25}\\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
E_{0,1}(z) & E_{0,2}(z) & E_{0,3}(z) & \cdots & E_{0, P-1}(z) & E_{0, P}(z)
\end{array}\right] .
$$

Suppose that $E_{p, q}(z)(p=0,1,2, \ldots, P, q=1,2, \ldots, P)$ is

$$
\begin{align*}
E_{p, q}(z) & =\frac{a_{p, q, 0}+a_{p, q, 1} z^{-1}+\cdots \cdots+a_{p, q, N_{p, q}} z^{-N_{p, q}}}{1+d_{p, q, 1} z^{-1}+d_{p, q, 2} z^{-2}+\cdots \cdots+d_{p, q, M_{p, q}} z^{-M_{p, q}}} z^{\tau_{p, q}} \\
p & =0,1,2, \ldots, P, \quad q=1,2, \ldots, P \tag{26}
\end{align*}
$$

To make $\mathbf{E}_{\mathbf{0}}, \mathbf{E}_{\mathbf{1}}, \ldots$, and $\mathbf{E}_{\mathbf{P}}$ able to be converted into the integer transform, from Theorem 1, The leading coefficients of $E_{1,1}(z), E_{2,2}(z), \ldots, E_{P, P}(z)$, and $E_{0, P}(z)$ should satisfy

$$
\begin{align*}
& a_{p, p, 0}=s_{p} 2^{k_{P}} \quad \text { for } p=1,2, \ldots, P, \quad a_{0, P, 0}=s_{0} 2^{k_{0}} \\
& s_{p}, s_{0}= \pm 1 \tag{27}
\end{align*}
$$

Then we discuss how to decompose $\mathbf{H}$ into the form as in (24) and convert it into a reversible integer transform.
(A) First, we discuss how to choose $\sigma$ in (24). Since

$$
\begin{gather*}
\operatorname{det}\left(\mathbf{E}_{\mathbf{p}}\right)=E_{p, p}(z), \quad \operatorname{det}\left(\mathbf{E}_{\mathbf{0}}\right)=E_{0, P}(z),  \tag{28}\\
\operatorname{det}(\mathbf{H})=\sigma^{P} E_{P, P}(z) E_{P-1, P-1}(z) \cdots E_{1,1}(z) E_{0, P}(z) . \tag{29}
\end{gather*}
$$

Note that, if $\Phi(z)=E_{P, P}(z) E_{P-1, P-1}(z) \ldots E_{1}(z) E_{0}(z)$, then the leading coefficient of $\Phi(z)$ is also a power of 2 . Therefore, if

$$
\begin{equation*}
\operatorname{det}(\mathbf{H})=\frac{a_{h, 0}+a_{h, 1} z^{-1}+a_{h, 2} z^{-2}+\cdots \cdots+a_{h, N_{h}} z^{-N_{h}}}{1+d_{h, 1} z^{-1}+d_{h, 2} z^{-2}+\cdots \cdots+d_{h, M_{h}} z^{-M_{h}}} z^{\tau_{h}}, \tag{30}
\end{equation*}
$$

to decompose $H(z)$ into (24), $\sigma$ should be chosen as:

$$
\begin{equation*}
\sigma=\left(\left|a_{h, 0}\right| 2^{-k_{\Phi}}\right)^{1 / P} \text { where } k_{\Phi} \text { is some integer. } \tag{31}
\end{equation*}
$$

(B) Suppose that

$$
\begin{equation*}
\boldsymbol{\Psi}=\sigma^{-1} \mathbf{H E}_{\mathbf{0}}^{-1} \tag{32}
\end{equation*}
$$

If $\boldsymbol{\Psi}$ can be decomposed as

$$
\begin{equation*}
\Psi=\mathbf{E}_{p} \mathbf{E}_{p-1} \mathbf{E}_{p-2} \ldots . \mathbf{E}_{1}, \tag{33}
\end{equation*}
$$

then the first row of $\mathbf{\Psi}$ is all the same as the first row of $\mathbf{E}_{\mathbf{1}}$, The first two rows of $\Psi$ are all the same as the first two rows of $\mathbf{E}_{2} \mathbf{E}_{\mathbf{1}}$, etc. Therefore, if $\boldsymbol{\Lambda}_{\mathrm{p}}$ is the upper-left $p \times p$ sub-matrix of $\boldsymbol{\Psi}$, then we can prove that

$$
\begin{equation*}
\operatorname{det}\left(\boldsymbol{\Lambda}_{\mathbf{p}}\right)=\prod_{q=1}^{p} E_{q, q}(z) \tag{34}
\end{equation*}
$$

From (27), the leading coefficient of $\operatorname{det}\left(\Lambda_{\mathrm{p}}\right)$ should be $\pm 2^{k}$. To satisfy it, $\mathbf{E}_{0}$ in (32) should be chosen properly. If

$$
\begin{equation*}
\mathbf{G}=\sigma^{-1} \mathbf{H} \tag{35}
\end{equation*}
$$

and $g_{p, q}(z)$ is the entry of $\mathbf{G}$ at the $p^{\text {th }}$ row and $q^{\text {th }}$ column, then the entry of $\mathbf{E}_{0}$ in (32) can be determined from

$$
\begin{align*}
E_{0,1}(z)= & \left(g_{1,1}(z)-f_{1}(z)\right) / g_{1, P}(z), \quad E_{0, P}(z)=1  \tag{36}\\
E_{0, p}(z)= & \left(\operatorname{det}\left(\mathbf{E}_{\mathbf{a}}-\mathbf{e}_{\mathbf{b}}^{\mathbf{T}} \mathbf{e}_{\mathbf{c}}\right)-f_{p}(z)\right) / \operatorname{det}\left(\mathbf{E}_{\mathbf{d}}\right)  \tag{37}\\
& \text { for } p=2,3, \ldots, P-1
\end{align*}
$$

where $f_{p}(z)$ are some filters whose leading coefficient is a power of 2, and

$$
\left.\begin{array}{l}
\mathbf{E}_{\mathbf{a}}=\left[\begin{array}{ccccc}
g_{1,1}(z) & g_{1,2}(z) & \cdots & g_{1, p-1}(z) & g_{1, p}(z) \\
g_{2,1}(z) & g_{2,2}(z) & \cdots & g_{2, p-1}(z) & g_{2, p}(z) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
g_{p, 1}(z) & g_{p, 2}(z) & \cdots & g_{p, p-1}(z) & g_{p, p}(z)
\end{array}\right], \\
\mathbf{e}_{\mathbf{b}}=\left[\begin{array}{llll}
g_{1, P}(z) & g_{2, P}(z) & \cdots & g_{p-1, P}(z)
\end{array} g_{p, P}(z)\right.
\end{array}\right], \begin{aligned}
& \mathbf{e}_{\mathbf{c}}=\left[\begin{array}{lllll}
E_{0,1}(z) & E_{0,2}(z) & \cdots & E_{0, p-1}(z) & 0
\end{array}\right], \\
& \mathbf{E}_{\mathbf{d}}=\left[\begin{array}{ccccc}
g_{1,1}(z) & g_{1,2}(z) & \cdots & g_{1, p-1}(z) & g_{1, P}(z) \\
g_{2,1}(z) & g_{2,2}(z) & \cdots & g_{2, p-1}(z) & g_{2, P}(z) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
g_{p, 1}(z) & g_{p, 2}(z) & \cdots & g_{p, p-1}(z) & g_{p, p}(z)
\end{array}\right] .
\end{aligned}
$$

(3) After $\mathbf{E}_{\mathbf{0}}$ is determined, calculate $\boldsymbol{\Psi}$ from (32). Then, we can determine $\mathbf{E}_{\mathbf{1}}, \mathbf{E}_{\mathbf{2}}, \ldots$, and $\mathbf{E}_{\mathbf{P}}$ iteratively from $\mathbf{\Psi}$. The first row of $\mathbf{E}_{1}$ is equal to that of $\boldsymbol{\Psi}$ and other rows of $\mathbf{E}_{1}$ are equal to those of the identity matrix. If

$$
\begin{equation*}
\Psi_{\mathrm{p}}=\Psi \mathbf{E}_{1}^{-1} \mathbf{E}_{2}^{-1} \ldots . \mathbf{E}_{\mathrm{p}-1}^{-1} \tag{42}
\end{equation*}
$$

then the $p^{\text {th }}$ row of $\mathbf{E}_{\mathbf{p}}$ is equal to that of $\boldsymbol{\Psi}_{\mathbf{p}}$ and other rows of $\mathbf{E}_{\mathbf{p}}$ are equal to those of the identity matrix.
(4) After the above process, all the components in (24) are derived successfully. Then, we can apply Theorem 1 to convert each part into a reversible integer transform.

Then, the forward integer multichannel transform corresponding to $\mathbf{H}$ can be implemented as:
(i) Set $u_{p}[n]=x_{p}[n]$ for $p=1,2, \ldots, P-1$,

$$
\begin{equation*}
u_{P}[n]=w_{0,1}[n]+w_{0,2}[n]+\cdots+w_{0, P-1}[n]+x_{P}[n] \tag{43}
\end{equation*}
$$

where $w_{0, q}[n]$ is calculated by the way as in (20).
(ii) $\operatorname{Set} p=1$.
(iii) $y_{p}[n]=w_{p, 1}[n]+w_{p, 2}[n]+\cdots+w_{p, P}[n]$
where $w_{p, q}[n]=s_{p, q, 0} v_{q}\left[n+\tau_{p, q}\right]+s_{p, q, 1} v_{q}\left[n+\tau_{p, q}-1\right]+\cdots \cdots+$
$s_{p, q, N_{p, q}} v_{q}\left[n+\tau_{p, q}-N_{p, q}\right]-c_{p, q, 1} w_{p, q}[n-1]-c_{p, q, 2} w_{p, q}[n-2]$
$-\cdots \cdots-c_{p, q, M_{p, q}} w_{p, q}\left[n-M_{p, q}\right]$,
$v_{q}[n]=u_{q}[n]$ if $q \geq p, \quad v_{q}[n]=y_{q}[n] \quad$ if $q<p$.
(iv) If $p \neq P$, set $p=p+1$ and return to Step (ii). If $p=P$, the process is completed.

It is clear that if $x_{p}[n](p \in[1, P])$ are sums of powers of 2, then the output $y_{p}[n](p \in[1, P])$ are also sums of powers of

2 and Constraint 1 in Section 1 is satisfied. The inverse integer multichannel transform can be implemented as:
(i) Set $p=P$.
(ii) $g_{p}[n]=y_{p}[n]-\sum_{\substack{k=1 \\ k \neq p}}^{P} w_{p, k}[n]$
where $w_{p, q}[n]$ is calculated from (45), but (46) is changed as

$$
\begin{equation*}
v_{q}[n]=u_{q}[n] \text { if } q>p, \quad v_{q}[n]=y_{q}[n] \quad \text { if } q \leq p . \tag{48}
\end{equation*}
$$

(iii) $u_{p}[n]=s_{p} 2^{-k_{p}}\left\{y_{p}\left[n-\tau_{p, p}\right]+c_{p, p, 1} y_{P}\left[n-\tau_{p, p}-1\right]+\cdots \cdots\right.$.

$$
\begin{align*}
& +c_{p, p, M_{p, p}} y_{p}\left[n-\tau_{p, p}-M_{p, p}\right]-s_{p, p, 1} u_{p}[n-1]- \\
& \left.s_{p, p, 2} x_{p}[n-2]-\cdots \cdots-s_{p, p, N_{p, p}} x_{p}\left[n-N_{p, p}\right]\right\} . \tag{49}
\end{align*}
$$

(iv) If $p \neq 1$, set $p=p-1$ and return to Step (ii). If $p=1$, continue to Step (v).
(v) Set $x_{p}[n]=u_{p}[n]$ for $p=1,2, \ldots, P-1$,

$$
\begin{equation*}
x_{P}[n]=u_{P}[n]-w_{0,1}[n]-w_{0,2}[n]-\cdots-w_{0, P-1}[n] \tag{50}
\end{equation*}
$$

where $w_{0, q}[n]$ is calculated by the way as in (20), but $x_{q}[n]$ in (20) is changed into $u_{q}[n]$.

We can prove one can perfectly recover the original signal $x_{p}[n]$ from $y_{p}[n]$ by the process of (47)-(50), i.e., Constraints 2 and 3 in Section 1 are satisfied. Therefore, the processes in (43)-(46) and (47)-(50) form a reversible integer multichannel transform pair.

## V. EXAMPLES

We give two examples as follows. The first example is a one-channel IIR smooth filter:

$$
\begin{equation*}
y[n]=(0.7)^{|n|} * x[n], \quad * \text { means the convolution. } \tag{51}
\end{equation*}
$$

To convert (52) into a reversible integer transform, first,

$$
\begin{aligned}
Y[z] & =\frac{1}{1-0.7 z^{-1}} X[z]+\frac{0.7 z}{1-0.7 z} X[z] \\
& =0.7286 \frac{-1}{1-2.1286 z^{-1}+z^{-2}} z^{-1} X[z] .
\end{aligned}
$$

Then, we follow the process shown in Section 2. If in (10)-(12) we choose $b_{1}=3$, then the reversible integer transform pair corresponding to (51) is:

$$
\begin{align*}
& t[n]=-x[n-1]-\frac{17}{8} t[n-1]+t[n-2],  \tag{52}\\
& x[n]=-t[n+1]+\frac{17}{8} t[n]-t[n-1] . \tag{53}
\end{align*}
$$

We can show that $t[n] \cong y[n] / 0.7286$ and Constraints 1,2 , and 3 in Section 1 are all satisfied. That is, (52) and (53) form a reversible integer transform pair.

Then, we give an example of a multichannel system. If there is a mutually interfered 2-channel system:

$$
\left[\begin{array}{l}
Y_{1}(z)  \tag{54}\\
Y_{2}(z)
\end{array}\right]=\left[\begin{array}{cc}
\frac{1}{1-0.6 z^{-1}} & \frac{0.3}{1-0.7 z^{-1}} \\
\frac{0.3}{1-0.6 z^{-1}} & \frac{1}{1-0.7 z^{-1}}
\end{array}\right]\left[\begin{array}{l}
X_{1}(z) \\
X_{2}(z)
\end{array}\right]
$$

Then, we can use (24) to decompose the system. From (28)(46), we obtain the following one-row matrices:

$$
\begin{align*}
& \mathbf{E}_{0}=\left[\begin{array}{cc}
1 & 0 \\
\frac{0.1535-0.1075 z^{-1}}{1-0.6 z^{-1}} & 1
\end{array}\right], \quad \mathbf{E}_{1}=\left[\begin{array}{cc}
\frac{1}{1-0.6 z^{-1}} & \frac{0.3145}{1-0.7 z^{-1}} \\
0 & 1
\end{array}\right], \\
& \mathbf{E}_{2}=\left[\begin{array}{cc}
1 & 0 \\
0.1535 & \frac{1}{1-0.7 z^{-1}}
\end{array}\right], \quad \sigma=0.9539 . \tag{55}
\end{align*}
$$

If in (13) $b_{1}=6$, we quantize $\mathbf{E}_{\mathbf{p}}$ into $\mathbf{T}_{\mathbf{p}}$ :

$$
\begin{align*}
& \mathbf{E}_{0}=\left[\begin{array}{cc}
1 & 0 \\
\frac{10}{64}-\frac{7}{64} z^{-1} & \\
1-38 z^{-1} / 64 & 1
\end{array}\right], \mathbf{E}_{1}=\left[\begin{array}{cc}
\frac{1}{1-\frac{38}{64} z^{-1}} & \frac{20 / 64}{1-\frac{45}{64} z^{-1}} \\
0 & 1
\end{array}\right], \\
& \mathbf{E}_{2}=\left[\begin{array}{cc}
1 & 0 \\
\frac{10}{64}-\frac{7}{64} z^{-1} & 1
\end{array}\right] . \tag{56}
\end{align*}
$$

Then, substituting the coefficients in (55) and (56) into (43)(46) and (47)-(50), we can convert the multichannel system in (44) into a reversible integer multichannel filter system.

## VI. Conclusion

The methods to convert one-channel or multi-channel IIR or FIR filter systems into reversible integer transforms are proposed in this paper. Our methods work successfully for any multichannel filter system if the determinant of its leading coefficients is nonzero. With the proposed method, one can recover the original signals from the outputs of multichannel systems perfectly without the error caused by truncation.

As the existing integer transforms, the proposed methods will be useful for communication, signal synthesis, MIMO system analysis, and lossy and lossless image compression.

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