Singularity-spreading phase unwrapping: Its basic idea and the influence of time and space discreteness on the dynamics

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***Abstract*—This paper discusses the background and the characteristic bases of singularity-spreading phase unwrapping (SSPU), a powerful and calculation-efficient method to unwrap two-dimensional phase map for generating digital elevation model (DEM) in interferometric synthetic aperture radar (InSAR). Since the phase value in InSAR directly corresponds to land height of earth surface, they should form a conservative field. In real observation, however, the interferogram often includes high-density singular points (SPs), that is, rotational points. Then a DEM cannot be obtained straightforward because of the SPs. Instead, we have to estimate the real land shape. Conventional estimation methods suffer from combinatorial explosion. However, our proposed SSPU shows almost constant low calculation cost independent of the SP number. It spreads the singularity, or rotation, outward radially. Apparently, the spreadability conflicts with the fact that the phase value change around a SP should be an integral multiple of 2π. But the spreading is possible because of the discreteness in space (pixels). That is, the difference-based calculation, instead of continuous differentiation, makes it possible to numerically realize the spreading. This paper revisits advantages and disadvantages in such discreteness useful in SSPU first. Then we discuss its extendability to other applications.**

I. Introduction

Phase singular point (SP) has been the most serious problem in interferometric synthetic aperture radar (InSAR) to generate digital elevation model (DEM). It has been investigated to realize a two-dimensional phase unwrapping (PU) method having low calculation-cost and high accuracy. Singularity-spreading phase unwrapping (SSPU) [2]–[5] is such a method, and very different from conventional ones before its proposal [6], [7]. Though the calculation cost is low, it unwraps a wide range of interferogram various in its resolution, incidence angle, landscape included in the data, and so on [8]–[16]. It attracts researchers in diverse fields such as optics [17]–[30], magnetic resonance imaging (MRI) and other biomedical imaging [31]–[33] as well as ultrasonic imaging [34], [35].

In satellite-borne and airborne SAR field, the origin of the SPs is going to be clarified in the near future [36], [37]. It was recently found that the phase distortion is highly correlated with polarization distortion, which suggests that one of the main origin is scattering itself, affecting both the phase and polarization simultaneously [38]–[40]. Then this harm can be removed by using the linear combination of polarization components. Another main origin is the interference among multiple scattering waves, which is presumed from the statistical characteristics [41], [42]. Other minor origins include thermal noise.

SSPU *diffuses* the singularity to vanish by utilizing the discreteness in space and time of interferogram data [1], [43]. It is essentially free from the combinatorial explosion problem, which is still a serious problem in most of the branch-cut type methods. Then the SSPU calculation time is very small independent of the SP number or density. In addition, the digital elevation model (DEM) obtained by SSPU has high fidelity, or high signal-to-noise ratio (SNR) when we know the ground truth.

On the other hand, however, SSPU sometimes suffered from the emerging harm of spatial or temporal discreteness in the past. To solve this problem, we proposed modifications to reduce the spatial discreteness [44], temporal discreteness [45] and temporal discreteness in combination with spatial limitation [1]. Evaluations demonstrated a SNR higher than that of the original SSPU.

In this paper, we review those methods briefly. Then we discuss the details of the relationship between the SSPU dynamics and the spatio-temporal discreteness.

II. “ Diffusion of singularity ” realized by the discreteness in space and time

A. Singularity spreading and the spatial discreteness

Fig. 1 is a schematic diagram to show the InSAR observation of earth surface from a satellite. The radar observes a point twice to obtain master and slave data, from which an interferogram is generated. In Fig. 1, R_{m} and R_{s} denote the distance between the satellite position from the observation point for the master and slave observations, B_{CT} and γ_{CT} are the length and the elevation angle of the baseline, i.e., the line between the master and slave positions, θ is the incidence angle, and H is the height of the point.

The interferogram phase $\phi(x, y)$ is obtained as the phase difference between the master and slave phase values, which is free from the phase changes at the scatterer, basically representing the wrapped height. Then the phase map $\phi(x, y)$ should be a conservative field. However, actual data contains

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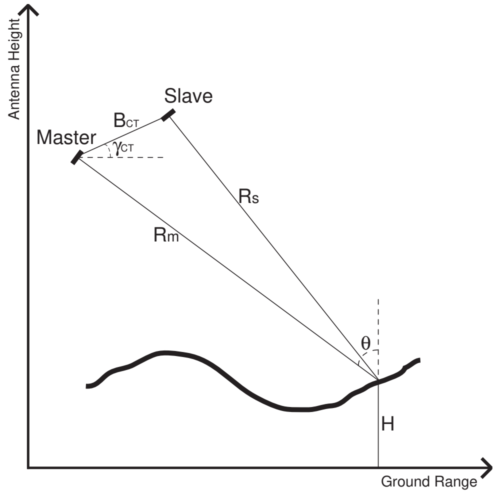


Fig. 1. Schematic diagram showing the PolInSAR observation from a satellite.

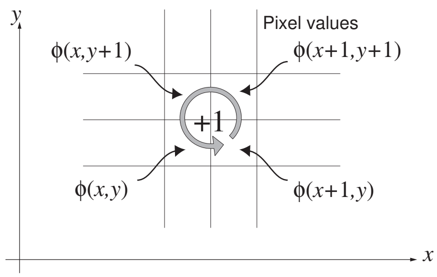


Fig. 2. A positive singular point (SP) in a interferogram phase map $\phi(x, y)$.

many SPs densely. Because of the SPs, we cannot obtain the height map in a straightforward manner but, instead, we have to estimate it.

Fig. 2 is an illustration of a positive SP. We call the four pixels constructing a SP a singular unit (SU). The pixel values in the SU are distorted, and should be restored in some manner.

Basic idea of the SSPU is explained as follows. If the phase distortion is compensated correctly, we will observe no rotation at the SP point. Hence, in the spatially finite difference space, we add inversely rotational components ($+\pi/4$ or $-\pi/4$ for each side) to the SU pixels as compensators as shown in Fig. 3. Then the SP at the SP point vanishes, and *quarter rotation* emerges around at the four sides in the spatially difference space, not in the original phase space. Only in the spatially difference space, which is discrete, the existence of such fractional rotation is allowed. In this sense, the SSPU dynamics is realized by the discreteness.

However, in precise observation of the SSPU dynamics, we find interleaved spreading [1] as shown below in Discussion Section (with Fig. 6). It is true that the SSPU spreads the singularity isotropically to four directions. However, in the repetition, the singularity is spread only on every other pixel, resulting in a checkerboard pattern. Depending on the relative location, the positive and negative SPs fail to cancel out but, instead, live long in the interleaved manner. This is the weak point in the original SSPU.

B. Less spatial-discreteness SSPU (Fully Isotropic SSPU)

The method, we named fully isotropic SSPU, solves this problem by reducing the spatial discreteness [1]. It realizes a quick cancellation of SP pairs with a shorter calculation time. The process is explained as follows. Fig. 4 presents how the fully isotropic SSPU spreads the singularity. In contrast with the four-directional spreading in the conventional SSPU, this method does the spreading into eight directions with the compensators expressed as

$$\begin{aligned}
\Delta\phi_{cx}^{(k)}\,(x,y) &= +2\pi\, R^{(k)}(x,y)\,/\,4 \\
\Delta\phi_{cx}^{(k)}\,(x\pm 1,y) &= +2\pi\, R^{(k)}(x,y)\,/\,24 \\
\Delta\phi_{cx}^{(k)}\,(x,y+1) &= -2\pi\, R^{(k)}(x,y)\,/\,4 \\
\Delta\phi_{cx}^{(k)}\,(x\pm 1,y+1) &= -2\pi\, R^{(k)}(x,y)\,/\,24 \\
\Delta\phi_{cy}^{(k)}\,(x,y) &= -2\pi\, R^{(k)}(x,y)\,/\,4 \\
\Delta\phi_{cy}^{(k)}\,(x,j\pm 1) &= -2\pi\, R^{(k)}(x,y)\,/\,24 \\
\Delta\phi_{cy}^{(k)}\,(x+1,y) &= +2\pi\, R^{(k)}(x,y)\,/\,4 \\
\Delta\phi_{cy}^{(k)}\,(x+1,j\pm 1) &= +2\pi\, R^{(k)}(x,y)\,/\,24 \qquad (1)
\end{aligned}$$

where $\Delta\phi_{cx}^{(k)}\,(x,y)$ and $\Delta\phi_{cy}^{(k)}\,(x,y)$ are compensators for phase differences in x and y directions, respectively, applied to position (x, y) at k-th iteration, while $\pm 2\pi R^{(k)}(x, y)$ is positive or negative residues remaining at k-th iteration. The addition of the diagonal directions mitigates the spatial discreteness in the spreading, and solves the interleave problem.

C. Less temporal-discreteness SSPU (Nonhollow SSPU)

Fig. 5 shows another method, namely the nonhollow SSPU. It reduces the temporal discreteness to solve the interleave problem [1]. In the conventional SSPU, a single spreading process moves the rotation ± 1 to the four neighbors with a weighting factor of $1/4$. Then the rotation at the original point becomes completely zero. This is the cause of the checkerboard pattern and the slow cancellation. We propose a spreading with smaller weights. We named this method nonhollow SSPU. The spreading process is represented with a spreading weight of $1/n \quad (1/n < 1/4)$ as

$$\begin{aligned}
\Delta\phi_{cx}^{(k)}\,(x,y+1) &= -\,2\pi\, R^{(k)}(x,y)\,/\,n \\
\Delta\phi_{cy}^{(k)}\,(x,y) &= -\,2\pi\, R^{(k)}(x,y)\,/\,n \\
\Delta\phi_{cx}^{(k)}\,(x,y) &= +\,2\pi\, R^{(k)}(x,y)\,/\,n \\
\Delta\phi_{cy}^{(k)}\,(x+1,y) &= +\,2\pi\, R^{(k)}(x,y)\,/\,n \qquad (2)
\end{aligned}$$

Because of the weight $1/n$ smaller than $1/4$, a part of the singularity $\pm(1 - 4R/n)$ remains at the original SP location, and the checkerboard pattern melts into alloy.

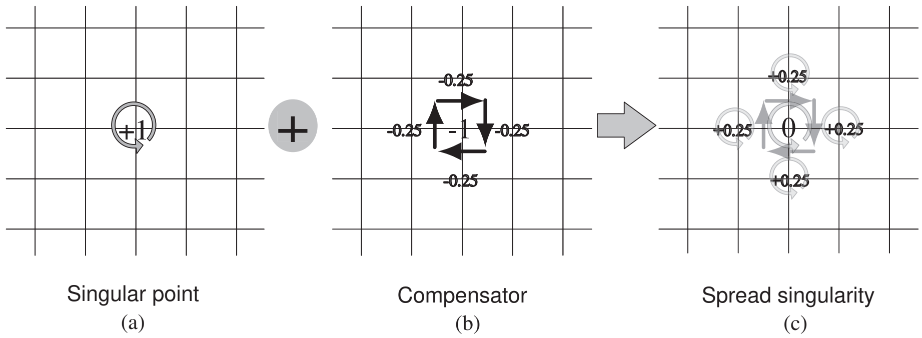


Fig. 3. Simple singularity-spreading phase-unwrapping (simple SSPU) process [6].

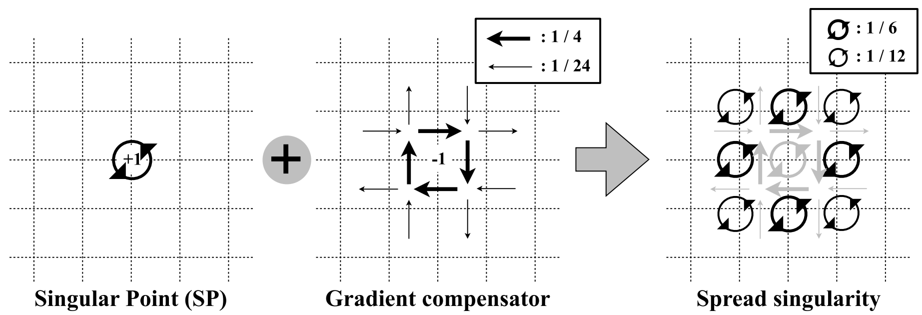


Fig. 4. Spreading process in fully-isotropic SSPU [1].

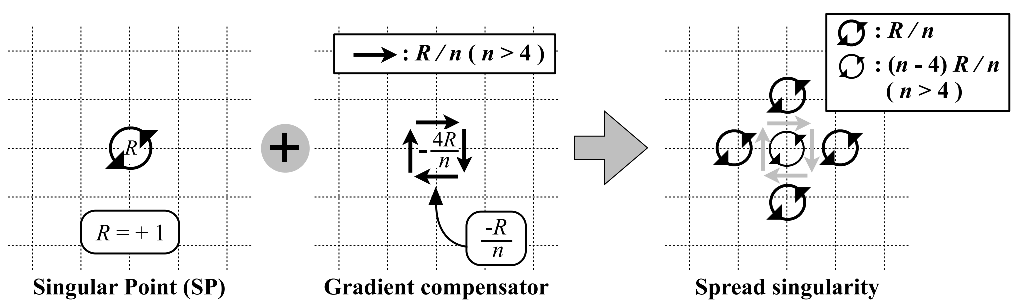


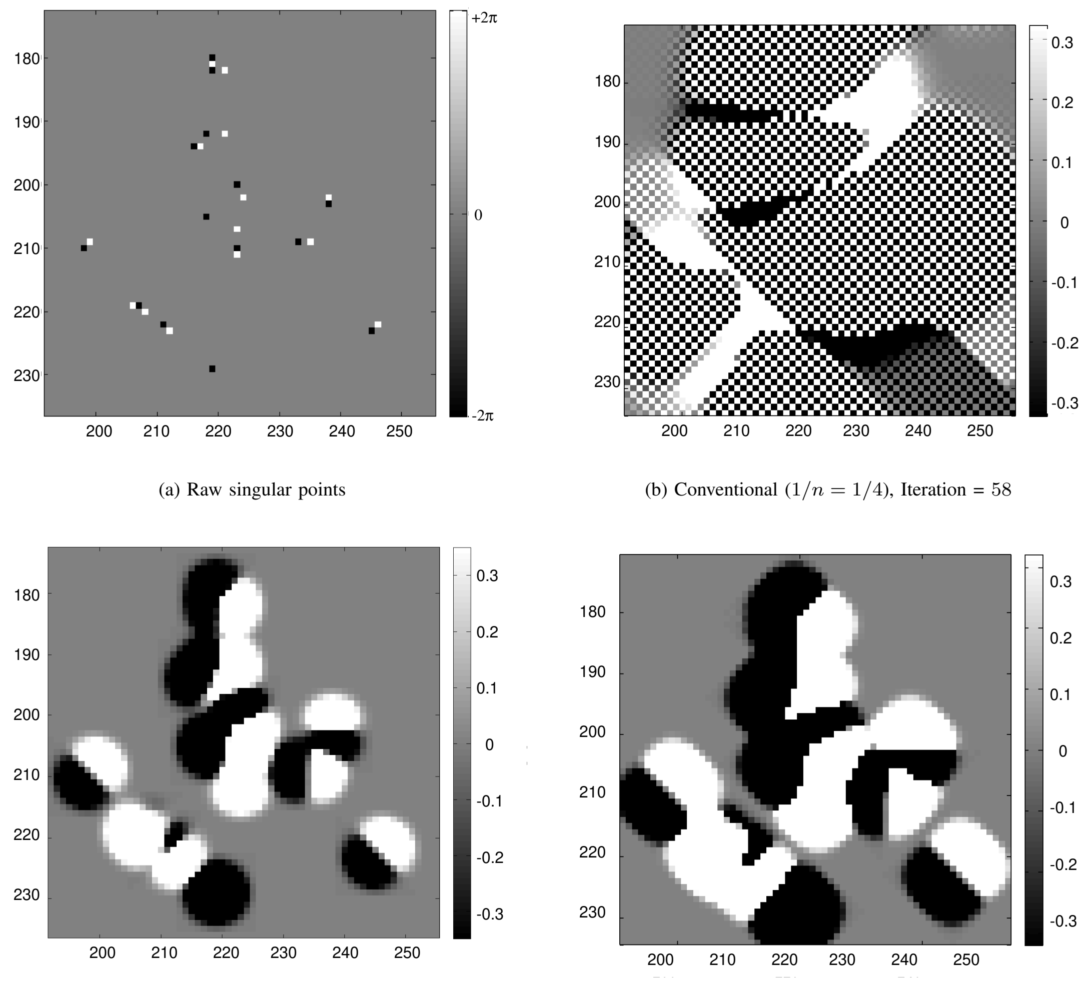
Fig. 5. Spreading process in nonhollow SSPU [1].

III. Profitable and adverse effects in the discreteness

Fig 6 presents (a) the original raw singular points (white: $+2\pi$, black: -2π) and the residual rotation maps (enhanced gray scale) in the course of spreading iterations when the rotation magnitude is reduced to such a level that $\max \|R^{(k)}\| \simeq \pi/10$, respectively, in results of (b)conventional SSPU, (c)fully-isotropic SSPU and (d) nonhollow SSPU, respectively. In Fig. 6(b), we find the checkerboard texture according to the even-odd interleaving spreading. Contrarily, both Figs. 6(c) and (d) show gentle and local spreading, which indicates effective combination of positive and negative SPs to vanish.

Consequently, though SSPU is based on the discreteness in space, the performance is maximized by mitigating the discreteness in some degree. This is an interesting point in the SSPU dynamics. Simultaneously, it is also attractive that SSPU generates a continuous DEM having high SNR in spite of the fact that the dynamics utilizes the discreteness essentially.

In von Neumann type computing, both space and time have to be discrete in reality. The fineness and roughness in the time discreteness can be determined arbitrarily depending on the calculation cost. In contrast, the spatial discreteness is fundamentally determined by the observation electronics. In the course of getting higher resolution of SAR, we should consider how to deal with *target scatterers* and how to create meanings to them. This situation is somewhat similar to



(a) Raw singular points

(b) Conventional ($1/n = 1/4$), Iteration = 58

(c) Fully-isotropic, Iteration = $k = 6$

(d) Nonhollow ($1/n = 1/5$), Iteration = $k = 11$

Fig. 6. (a)Raw singular points (white: positive, black: negative) and residual rotation maps (enhanced gray scale) in the course of spreading iterations when the rotation magnitude is reduced to such a level that $\max \|R^{(k)}\| \simeq \pi/10$ in (b)conventional SSPU, (c)fully-isotropic SSPU and (d) nonhollow SSPU, respectively. (Modified from Ref. [1].)

the relationship between usual InSAR and persistent-scatterer InSAR.

IV. Summary

This paper discussed the essentially profitable and adverse effects in the spatial and temporal discreteness in SSPU dynamics. Though SSPU does not include physics in its basis, it shows a high performance in generation DEM in its accuracy and calculation cost. Interesting points have been argued mainly by focusing on the utilization of the discreteness and the high performance obtained by the mitigation of the discreteness.

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