Energy Efficiency Competition for Visible Light Communication under Illumination Constraint

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***Abstract*—This paper concentrates on energy efficient visible light communication (VLC) under illumination constraint. To facilitate the implementation in practice, we investigate the problem from a distributed perspective that each user attempts to maximize its own energy efficiency (EE) of VLC while subject to the shared constraint on the illumination. The competition is formulated as a generalized Nash equilibrium problem (GNEP), and the optimal transmission strategy is proposed for each individual user by analyzing the game equilibrium. Also, the numerical results are provided to corroborate our theoretical findings and demonstrate the superiority of our proposal in terms of EE as compared with conventional approaches.**

I. Introduction

Visible light communication (VLC) has emerged as a promising solution to support high-data-rate transmissions in indoor wireless environment, where there occurs 70% of mobile Internet traffic [1]. VLC exploits the visible light spectrum for transmissions, which provides a powerful complement for conventional radio frequency communications [2], [3]. The light emitting diodes (LEDs) used in VLC are capable to provide illumination and communication simultaneously. As such, VLC has the advantage of low-cost equipment, low-power transmission, and vast bandwidth, and thus has attracted an increasing research interest [4].

However, the main research efforts on VLC have been devoted to throughput oriented analysis and strategy design [5]–[10], while the energy related issues are relatively less addressed. Despite the fact that the LEDs are indeed less energy-consuming as compared with conventional lighting sources, there is still significant space to further save the energy when we jointly consider the illumination and communication. In this respect, there have emerged a few recent results on energy efficient VLC [11]–[13], which are all based on the centralized optimization. The centralized methods usually are of high complexity and require additional signaling overhead, which may not always be capable to adapt to the changing environment. For the actual VLC system, the VLC access points (APs) are usually immobilized, while the positions of the user equipment (UE) and the angles of the receiving photo diode (PD) are likely to change constantly. Those changes will instantly affect the VLC channels as VLC is highly dependent on the line-of-sight (LOS) links [14]. As such, VLC is desired to achieve energy-efficient transmissions while capable for quick adaptation. In this regard, the distributed strategy design, which allows the individual communication pair to determine its transmission independently, has the inherent advantage of ease of implementation and lower signaling overhead [15]–[17]. Moreover, in the existing works [5]–[7], [11]–[13], the mutual interference issue in VLC has not been sufficiently addressed. Yet as we know, the avoidance or mitigation of the interference require additional resources and coordination, which may not be economical or even feasible. In this regard, we need to actively taken into account the mutual interference for distributed decision-making at individual user to further ease the implementation.

On the other hand, the preliminary function of the LEDs, besides the communication, is to provide illumination [6], [11], [18]–[21]. In order to protect the human eyes with comfortable enjoying the indoor environment, the illumination should be properly constrained. Consequently, it is essential to jointly consider the communication and illumination. As the illumination results from the aggregated effect from all VLC APs, their transmission behavior should be coordinated to satisfy the shared illumination constraint. Therefore, although the distributed approach can be sufficient for the individual energy efficiency (EE) optimization, we still need effective coordination among them in the respect of illumination control.

Targeting at the aforementioned issues, we in this paper investigate the distributed energy efficient VLC under the illumination constraint. In particular, we consider that each VLC transmission pair attempts to maximize its own EE while subject to the shared constraint regarding illumination. The generalized Nash equilibrium problem (GNEP) model is adopted to analyze their competition. Due to the inherent difficulties of the GNEP, we introduce the pricing techniques to tackle the shared illumination constraint and reformulate

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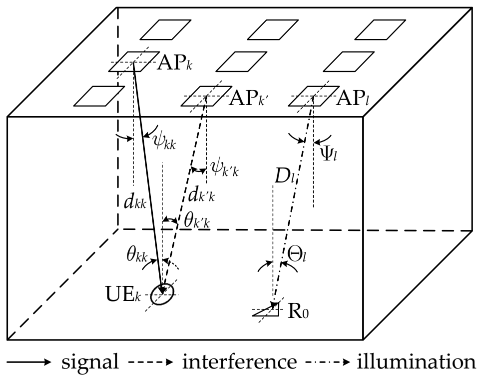

Fig. 1. System model.

the GNEP as a conventional NEP. The game equilibrium is analyzed with the distributed transmission strategy proposed for each VLC pair to effectively maximize the its own EE while guaranteeing the shared illumination constraint.

The rest of this paper is organized as follows. In Sec. II, we introduce the system model of VLC communications. In Sec. III, we addresses the game formulation of the energy efficient VLC communications with distributed algorithm design. In Sec. IV, we provides the simulation results and the paper is concluded in Sec. V.

II. System Model

We consider an indoor environment where there are L VLC APs, given by $\mathcal{L} = \{1, 2, \cdots, L\}$, uniformly mounted on the ceiling to provide the illumination and communication. Meanwhile, there are K UEs, dented by $\mathcal{K} = \{1, 2, \cdots, K\}$, intending for communication services. The system is illustrated in Fig. 1. In this paper, we consider the case that the number of APs is no less than that of UEs, and thus each AP has at most one UE in service. In this regard, each UE is associated with the nearest AP available in its proximity, since the VLC mainly counts on the LOS link. Without loss of generality, we can denote the AP-UE pairs[1] as $\mathcal{K}$, and thus the APs in $\mathcal{L} \backslash \mathcal{K}$ only provide illumination. For each AP-$l \in \mathcal{L}$, we denote the power from direct current (DC) as q_l, which acts as the carrier and is of predefined fixed value. Meanwhile, the power from alternating current (AC) for VLC is denoted by p_l and bounded within region $\mathcal{P}_l^{\circ} = [0, p_l^{\max}]$.

Illumination is the primary function of the LEDs. To effectively track the achieved illumination level, we here specially consider a reference point, denoted by R_0. This reference point may correspond to the coach at home, or the desk in an office, where the proper illumination level should be guaranteed. Note this model can be easily extended to more the general cases such as average illumination, minimum or maximum illumination, etc. Considering the Lambert radiation characteristic, we can model the channel gain from AP-l to the reference point as

$$g_l = \frac{m+1}{2\pi D_l^2} \cos^m\left(\Theta_l\right) \cos\left(\Psi_l\right), \tag{1}$$

where m is the Lambertian order by $m = \log(2)/\log\left(\cos\left(\phi_{1/2}\right)\right)$ with $\phi_{1/2}$ being the semi-angle at half-power, D_l is the distance between the AP-l and reference point, Θ_l and Ψ_l are the angle of radiance and incidence, respectively. Then, the illuminance at the reference point can be calculated by aggregating the power of AC and DC from all APs, given as

$$I_0 = \sum\nolimits_{l \in \mathcal{L}} \rho\left(p_l + q_l\right) g_l, \tag{2}$$

where ρ is the luminosity efficacy[2]. As the power for VLC acts as additional sources for lighting, we present an upper bound for the illumination at the reference point to protect human eyes, specified as $I_0 \le I^{\text{th}}$. Besides the intention for eye health, this upper bound also helps to save the energy. Note we here only explicitly consider the upper bound, the similar analysis can be conducted regarding the lower bound, and we omit the detailed analysis for space limitation.

VLC is conducted at all the users in $\mathcal{K}$, whose transmissions reuse the same frequency band and thus constitute the interference model. The link gain from AP-k' to UE-k is given as

$$h_{k'k} = \begin{cases} \frac{(m+1)A_k}{2\pi d_{k'k}^2} \cos^m\left(\theta_{k'k}\right) \cos\left(\psi_{k'k}\right) \frac{n^2}{\sin^2\left(\psi_k^{\mathrm{c}}\right)} G\left(\psi_{k'k}\right), \\ \qquad\qquad\qquad \text{if} \quad 0 \le \psi_{k'k} \le \psi_k^{\mathrm{c}}, \\ 0, \qquad\qquad\qquad \text{otherwise}, \end{cases} \tag{3}$$

where A_k is the area of receiving PD, n is the refractive index, ψ_k^{c} is the half field of view, $G\left(\psi_{k'k}\right)$ is the gain of optical filter, and other parameters are similarly defined as those in (1). Then, the achieved transmission rate can be calculated based on the Shannon-Hartley theorem as

$$r_k = \omega \log\left(1 + \frac{p_k h_{kk}}{\sum_{k' \in \mathcal{K}\backslash\{k\}} p_{k'} h_{k'k} + \sigma_0^2}\right), \tag{4}$$

at user-k, where ω is the bandwidth and σ_0^2 denotes the power of noise. With the achieved transmission rate, the EE is defined as the bits delivered by unit energy, given as

$$\eta_k = \frac{r_k}{p_k + q_k}. \tag{5}$$

Note the energy consumption for VLC in (5) concerns both DC and AC power, with the former acting as the carrier for the latter.

III. Game Formulation and Algorithm Design

In a distributed manner, each user involved in VLC aims at maximizing its own EE. Since the transmission at different users induces interference and thus affects each other, such an interest-conflicting scenario is modeled with game theory.

[1]Hereinafter, we refer to the VLC AP-UE pair as user for simplicity.

[2]The channel gains corresponding to p_l and q_l are actually of slight difference. But here we ignore the difference as we mainly focus on the communications.

However, in our considered problem, the EE competition needs to consider an outside constraint regarding illumination as $I_0 \leq I^{\text{th}}$. As such, the feasible region of at user-k is actually

$$\mathcal{P}_k\left(\boldsymbol{p}_{-k}\right) = \left\{ p_k \in \mathcal{P}_k^{\circ} \,\middle|\, \sum_{k' \in \mathcal{K}} \rho p_{k'} g_{k'} \leq I^{\text{th}} - \sum_{l \in \mathcal{L}} \rho q_l g_l \right\}, \tag{6}$$

where $-k$ denotes all users in $\mathcal{K}$ other than user-k and the illumination constraint is reorganized in (6) as $p_l = 0$, $\forall\, l \in \mathcal{L} \backslash \mathcal{K}$. Then, the individual problem at user-k can be given as

$$\max_{p_k} \;\; \eta_k \qquad \text{s.t.} \quad p_k \in \mathcal{P}_k, \tag{7}$$

from which we can see that both the objective function and feasible region at one user depend on the strategies of all other users. In this regard, the competition in the form of (7) goes beyond the conventional Nash sense, and constitutes a GNEP[3] [22].

The solution to the GNEP is noted as GNE. To tackle the GNEP, we first need to confirm the existence of GNE. Revisit the problem in (7), we can easily prove that EE as the objective function is quasi-concave with respect to its own transmit power. Meanwhile, the feasible region is defined by a set of linear inequalities, and thus is always convex, regardless of the strategies of other users. Those properties hold at all users, which further guarantees the existence of the GNE[4].

Then, we concentrate on the derivation of the GNE. As we can readily notice, the main difficulties regarding the GNEP is that the feasible region is not fixed, which is due to the shared illumination constraint. To solve the problem effectively, we can tackle the illumination constraint within the objective function through pricing technique. By pricing the illumination condition, the newly obtained objective function for user-k is given as[5]

$$u_k = \eta_k - \lambda \sum\nolimits_{k' \in \mathcal{K}} \rho p_{k'} g_{k'}, \tag{8}$$

where λ is the pricing coefficient that satisfies

$$0 \leq \lambda \perp I^{\text{th}} - \sum\nolimits_{l \in \mathcal{L}} \rho q_l g_l - \sum\nolimits_{k' \in \mathcal{K}} \rho p_{k'} g_{k'} \geq 0, \tag{9}$$

with $0 \leq x \perp y \geq 0$ indicating $x, y \geq 0$ and $x \cdot y = 0$. Note the pricing coefficient λ is globally defined, and thus identical at all users. Now we fix the pricing coefficient and consider the competition among the users with the priced objective function. As the shared illumination constraint is removed, the problem at user-k can be given as

$$\max_{p_k} \;\; u_k \qquad \text{s.t.} \quad p_k \in \mathcal{P}_k^{\circ}. \tag{10}$$

Then, we can see that the GNEP in (7) now reduced to the NEP in (10). For the NEP, we can confirm the existence of the NE as the solution due to the fact that u_k is quasi-concave with respect to p_k, and $\mathcal{P}_k^{\circ}$ is convex, for all $k \in \mathcal{K}$. To obtain the NE, we can adopt the best-response iteration. Specifically, the best-response strategy for user-k can be calculated by nulling the first-order derivative of u_k while fixing the transmit power of all others. We denote the best-response function as $p_k^\star = \mathsf{BR}_k\left(\boldsymbol{p}_{-k}\right)$, which is a function of the strategies of all others that satisfies

$$\begin{aligned} &\frac{\omega h_{kk}}{p_k^\star h_{kk} + \sum_{k' \in \mathcal{K} \backslash \{k\}} p_{k'} h_{k'k} + \sigma_0^2} \\ &= \eta_k\left(p_k^\star, \boldsymbol{p}_{-k}\right) + \lambda \rho g_k\left(p_k^\star + q_k\right). \end{aligned} \tag{11}$$

Although we cannot obtain an analytical expression for BR_k, we can calculate it efficiently based on (11) with bi-directional search. With the assistance of the best-response function, the NE for the NEP in (10) can be obtained as the fixed point of the best-response function as $p_k^\star = \mathsf{BR}_k(\boldsymbol{p}_{-k}^\star)$, $\forall\, k \in \mathcal{K}$. On obtaining the NE with current pricing coefficient, we can employ the subgradient method to update the price, specified as

$$\lambda \leftarrow \left[\lambda - \epsilon\left(I^{\text{th}} - \sum\nolimits_{l \in \mathcal{L}} \rho q_l g_l - \sum\nolimits_{k' \in \mathcal{K}} \rho p_{k'} g_{k'}\right)\right]^+, \tag{12}$$

where $\epsilon \geq 0$ is the step size and $(\,\cdot\,)^+ = \max\{\,\cdot\,, 0\}$. Through the constraint in (9), we can see that when the illumination constraint is strictly satisfied, the price vanishes and the NEP becomes equivalent to the original GNEP. Otherwise, the price becomes higher until the illumination constraint is satisfied with equality. In this regard, we can see that when the iteration in (12) is sufficiently updated such that λ is sufficiently close to zero, the obtained NE will sufficiently approach the GNE of the EE competition with shared illumination constraint satisfied.

Based on the preceding analysis, we know that the GNEP can be solved with two-tier iterations. For the inner iterations, we fix the shared price coefficient and update the power allocation at each user by conducting the best-response strategy. For the outer iterations, we update the pricing coefficient through the subgradient method. The GNE can be obtained when the convergence is arrived through the iterations. In this respect, the out iteration requires the information exchange among the users to coordinate the pricing coefficient. In contrast, the users can conduct the best-response strategy independently at their own for the inner iterations. The algorithm to solve the GNEP can be summarized as Algorithm 1.

IV. Simulation Results

Based on the preceding analytical work, we then conduct the numerical results to evaluate the performance under our proposals. We consider an indoor environment as illustrated by Fig. 1 with the main simulation parameters are shown in Table I. The positions of mounted APs on the room ceiling and the reference point for illumination on the floor are shown as the vertical view in Fig. 2.

As we propose to reach the GNE though iterations, we explicitly show an iterative process as Fig. 3, which concerns

[3] For the conventional NEP, the players affect each other through the objective function, while the feasible region at each player is independent. While for the GNEP, both the objective and feasible region are affected by other players.

[4] The detailed proof is omitted here for space limitation, the interested reader can refer to [17], [22] for related discussions.

[5] Originally, we need to incorporate the illumination constraint in the objective function as $u_k = \eta_k - \lambda\left(\sum_{k' \in \mathcal{K}} \rho p_{k'} g_{k'} + \sum_{l \in \mathcal{L}} \rho q_l g_l - I^{\text{th}}\right)$. But we can safely employ the representation in (8) as the sum of neglected items is constant.

Algorithm 1 EE competition with illumination constraint

Initialization;
repeat
  repeat
    for $k \in \mathcal{K}$ **do**
      Strategy update at user-k according to (11);
    end for
  until NE obtained for current pricing coefficient;
  Pricing coefficient update according to (12);
until Convergence achieved.

TABLE I
SIMULATION PARAMETERS

Parameter	Value
Room size	6 m×6 m×3 m
Number of APs	9
Coordinates of R_0	(2, 2, 0)
DC power	4 W
Maximum AC power	4 W
Semi-angle at half power	70°
Luminosity efficacy	683 lm/W
Area of PD	1 cm^2
Refractive index	1.5
Field of view	120°
Optical filter gain	1
Power of background noise	−100 dBm
Bandwidth	20 MHz

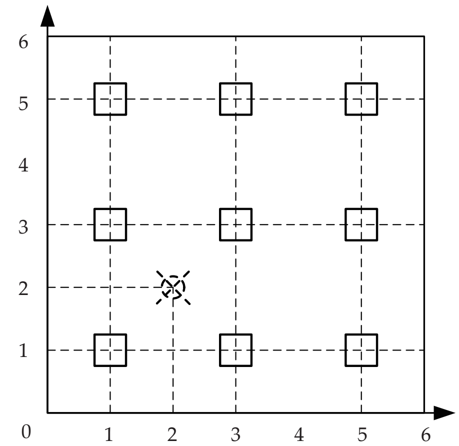

Fig. 2. Topology of the VLC APs in the indoor environment.

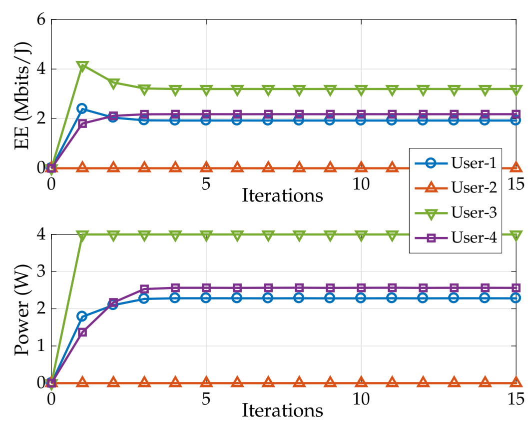

Fig. 3. Convergence of the distributed EE competition.

4 active VLC users. In this figure, we can see that the convergence to the GNE can be obtained rather efficiently, as only a few iterations are required. During this process, we can see that the power allocation and EE of the users fluctuate, as a result of their competing transmission behavior. At the convergence, we can see that User-2 has been dropped out of the competition, may due to poor link quality or high interference. While User-3 adopts full-power transmission which is also of the highest EE, indicating that its experienced interference is quit slight.

In Fig. 4, we depict the allocated power and illumination distribution for two typical cases at the GNE with 4 VLC users. In particular, for the case with the upper two subfigures, we can see that the APs involved in VLC are relatively far from the reference point. Then the illuminance at the reference point is 378 lux, which is smaller than the threshold, yet still satisfies the constraint. For the case of lower two subfigures, the illuminance at the reference point is rigorously 400 lux. Consequently, we can see that the illumination constraint can be always guaranteed during the EE competition.

In Fig. 5, we demonstrate the performance with respect to the number of active VLC users. For the EE, we can see that when there are more users, the interference in the system increases, and thus the average EE is decreased. Compared with the conventional strategy to maximize the transmission rate under the same illumination constraint, the EE is evidently improved under our proposal. Moreover, we can see that the illumination constraint significantly affects the transmission behavior of users. When the illumination constraint is ignored, more users are admitted to joint the competition and thus the average EE will be degraded due to the increased interference. For the achieved transmission rate, we can see that the distributed rate maximization, naturally, achieves the highest average rate, which dominates our approach. The preceding observations can be further explained by the figure on the average transmit power. In this figure, we can see that the average transmit power under our proposal with illumination constraint is the lowest. In contrast, when the illumination constraint is inactive, the competition results in the nearly full-power transmission at all users. Since the actually achieved illuminance depends on the power, we can see correspondingly that if there is no illumination constraint, the illumination keeps increasing with more users joining in the communication. For our proposed scheme, we can see that the achieved illuminance is slightly lower than the threshold.

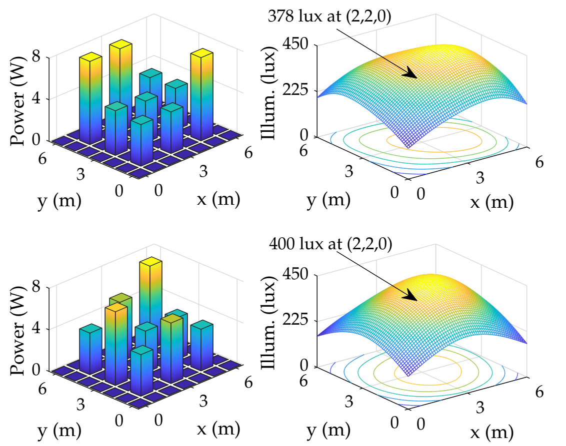

Fig. 4. The power for VLC at different APs and the illumination distribution in the room.

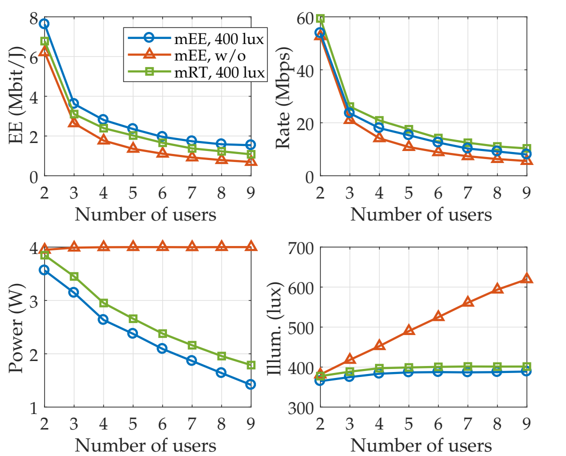

Fig. 5. Performance in terms of EE, rate, power, and illuminance averaged over active users against the number of users. "mEE, 400 lux" indicates to maximize EE with illuminance threshold at 400 lux; "mRT, 400 lux" to maximize transmission rate with illuminance threshold at 400 lux; "mEE, w/o" to maximize EE without illumination constraint.

In contrast, as the rate maximization inclines maximum power transmissions, the distributed rate maximization results in the illuminance always being equal to the threshold.

V. Conclusion

To summarize, we in this paper investigate the distributed energy efficient VLC under the shared illumination constraint. The problem is solved based on the GNEP model and the distributed algorithm is proposed based on the analysis on the equilibrium. The numerical results demonstrate that our proposed scheme converges to the equilibrium effectively. Moreover, it achieves the maximum EE while guaranteeing the illumination reliably.

References

[1] S. Chen and J. Zhao, "The requirements, challenges, and technologies for 5G of terrestrial mobile telecommunication," *IEEE Commun. Mag.*, vol. 52, no. 5, pp. 36–43, May 2014.

[2] A. T. Hussein and J. M. H. Elmirghani, "Mobile multi-gigabit visible light communication system in realistic indoor environment," *J. Lightwave Technol.*, vol. 33, no. 15, pp. 3293–3307, Aug. 2015.

[3] A. Jovicic, J. Li, and T. Richardson, "Visible light communication: opportunities, challenges and the path to market," *IEEE Commun. Mag.*, vol. 51, no. 12, pp. 26–32, Dec. 2013.

[4] P. H. Pathak, X. Feng, P. Hu, and P. Mohapatra, "Visible light communication, networking, and sensing: a survey, potential and challenges," *IEEE Commun. Surveys Tuts.*, vol. 17, no. 4, pp. 2047–2077, 4th quarter 2015.

[5] B. Hussain, X. Li, F. Che, C. P. Yue, and L. Wu, "Visible light communication system design and link budget analysis," *J. Lightwave Technol.*, vol. 33, no. 24, pp. 5201–5209, Dec. 2015.

[6] R. Jiang, Z. Wang, Q. Wang, and L. Dai, "Multi-user sum-rate optimization for visible light communications with lighting constraints," *J. Lightwave Technol.*, vol. 34, no. 16, pp. 3943–3952, Aug. 2016.

[7] S. Zhao and X. Ma, "A spectral-efficient transmission scheme for dimmable visible light communication systems," *J. Lightwave Technol.*, vol. 35, no. 17, pp. 3801–3809, Sep. 2017.

[8] D. A. Basnayaka and H. Haas, "Design and analysis of a hybrid radio frequency and visible light communication system," *IEEE Trans. Commun.*, vol. 65, no. 10, pp. 4334–4347, Oct. 2017.

[9] O. Narmanlioglu, R. C. Kizilirmak, T. Baykas, and M. Uysal, "Link adaptation for MIMO OFDM visible light communication systems," *IEEE Access*, vol. 5, pp. 26 006–26 014, 2017.

[10] H. Zhang, W. Ding, J. Song, and Z. Han, "A hierarchical game approach for visible light communication and d2d heterogeneous network," in *Proc. IEEE Global Communications Conference (GLOBECOM)*, Washington, DC, Dec. 2016, pp. 1–6.

[11] I. Din and H. Kim, "Energy-efficient brightness control and data transmission for visible light communication," *IEEE Photon. Technol. Letts.*, vol. 26, no. 8, pp. 781–784, Apr. 2014.

[12] R. Zhang, H. Claussen, H. Haas, and L. Hanzo, "Energy efficient visible light communications relying on amorphous cells," *IEEE J. Sel. Areas Commun.*, vol. 34, no. 4, pp. 894–906, Apr. 2016.

[13] M. Kashef, M. Ismail, M. Abdallah, K. A. Qaraqe, and E. Serpedin, "Energy efficient resource allocation for mixed RF/VLC heterogeneous wireless networks," *IEEE J. Sel. Areas Commun.*, vol. 34, no. 4, pp. 883–893, Apr. 2016.

[14] P. Chvojka, S. Zvanovec, P. A. Haigh, and Z. Ghassemlooy, "Channel characteristics of visible light communications within dynamic indoor environment," *J. Lightwave Technol.*, vol. 33, no. 9, pp. 1719–1725, May 2015.

[15] Y. Sun, M. Peng, and H. V. Poor, "A distributed approach to improving spectral efficiency in uplink device-to-device enabled cloud radio access networks," *IEEE Trans. Commun.*, 2018, to be published.

[16] X. Tang, P. Ren, and Z. Han, "Hierarchical competition as equilibrium program with equilibrium constraints towards security-enhanced wireless networks," *IEEE J. Sel. Areas Commun.*, 2018, to be published.

[17] Z. Han, D. Niyato, W. Saad, T. Başar, and A. Hjørungnes, *Game Theory in Wireless and Communication Networks*. Cambridge, UK: Cambridge University Press, 2011.

[18] A. Tsiatmas, C. P. M. J. Baggen, F. M. J. Willems, J. P. M. G. Linnartz, and J. W. M. Bergmans, "An illumination perspective on visible light communications," *IEEE Commun. Mag.*, vol. 52, no. 7, pp. 64–71, Jul. 2014.

[19] J. Gancarz, H. Elgala, and T. D. C. Little, "Impact of lighting requirements on VLC systems," *IEEE Commun. Mag.*, vol. 51, no. 12, pp. 34–41, Dec. 2013.

[20] S. Li, A. Pandharipande, and F. M. J. Willems, "Unidirectional visible light communication and illumination with LEDs," *IEEE Sensors J.*, vol. 16, no. 23, pp. 8617–8626, Dec. 2016.

[21] M. Morales-Cspedes, M. C. Paredes-Paredes, A. G. Armada, and L. Vandendorpe, "Aligning the light without channel state information for visible light communications," *IEEE J. Sel. Areas Commun.*, vol. 36, no. 1, pp. 91–105, Jan. 2018.

[22] J.-S. Pang and M. Fukushima, "Quasi-variational inequalities, generalized nash equilibria, and multi-leader-follower games," *Comput. Manag. Sci.*, vol. 6, no. 3, pp. 373–375, 2009.