Performance Evaluation of Phase-Only Correlation Functions from the Viewpoint of Correlation Filters

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***Abstract*—This paper proposes performance evaluation of phase-only correlation (POC) functions using signal-to-noise ratio (SNR) and peak-to correlation energy (PCE) from the viewpoint of correlation filters. Correlation functions can be thought as the output from the correlation filters. Maximizing SNR leads to matched filters, whereas maximizing PCE results in the inverse filters. We also derive the general expressions of SNR and PCE of the POC functions based on directional statistics. SNR is expressed by simple fractional function of circular variance. PCE is simply given by squared peak value of the POC functions, and its expectation can be expressed in terms of circular variance.**

I. Introduction

Phase-only correlation (POC) functions have been widely used for evaluating similarity between two signals. They have been applied for in many fields, such as image registration [1]–[3], pattern recognition [4], [5], motion estimation [6], [7], and so on. In order to clarify the effects of stochastic phase-spectrum differences on the POC functions, our group proposed statistical analysis of the POC functions with stochastic phase-spectrum differences [8]–[10]. In Ref. [9], we proposed statistical analysis method for the POC functions with stochastic phase-spectrum differences based on *directional statistics*. We assume phase-spectrum differences between two signals to be random variables with some *circular* probability distributions.

On the other hand, correlation functions can be thought as the output from the correlation filters. In order to evaluate performance of the correlation filters, signal-to-noise ratio (SNR) and peak-to-correlation energy (PCE) are commonly used for correlation performance measures [5]. SNR is the ratio of the square of average correlation peak to its variance. PCE is a peak sharpness measure of correlation functions. We have considered that these performance measures are applicable for also the POC functions.

This paper proposes performance evaluation of the POC functions using SNR and PCE from the viewpoint of correlation filters. We derive the general expressions of SNR and PCE of the POC functions as correlation performance measures. SNR is expressed by simple fractional function of circular variance. PCE is simply given by squared peak value of the POC function, and its expectation can be expressed in terms of circular variance.

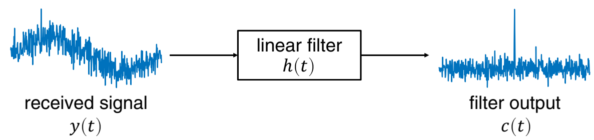

Fig. 1. Basic idea for signal detection problem by linear filter.

II. Correlation Performance Measures Based on Correlation Filters

This section gives preliminaries about correlation filters. SNR and PCE are commonly used correlation performance measures [5].

A. Correlation Filters

In the field of pattern recognition, correlation methods are commonly used for detecting desired signals from received signal.

Let $x(t)$ and $y(t)$ denote the reference signal and the received signal, respectively. We assume that the received signal $y(t)$ is corrupted by additive noise $v(t)$, and we judge the absence or presence of the reference signal $x(t)$ from the received signal $y(n)$. We have the following two hypotheses:

$$H_0 : y(t) = v(t) \tag{1}$$

$$H_1 : y(t) = x(t) + v(t) \tag{2}$$

where H_0 and H_1 denote the absence and presence of the reference signal, respectively. The noise $v(t)$ is assumed to be random process with zero mean and power spectral density (PSD) $P_v(f)$. We select between the two hypotheses from the received signal $y(t)$ with knowledge of $x(t)$ and $P_v(f)$. Basic idea for this signal detection problem by linear filter is shown in Fig. 1. The filter output $c(t)$ is given by

$$c(t) = \int_{-\infty}^{\infty} h(\tau) y(t-\tau) d\tau \tag{3}$$

where $h(t)$ is the impulse response of the linear filter. From the output of the linear filter $c(t)$, we have to select the hypothesis yielding the lowest error probability.

B. Signal-to-Noise Ratio (SNR)

Signal-to-Noise Ratio (SNR) of the filter output $c(t)$ is defined as follows:

$$\mathrm{SNR} = \frac{|E[c_{\max}]|^2}{\mathrm{Var}[c_{\max}]} \tag{4}$$

where $c_{\max}$ is the maximum value of $c(t)$. We assume that the filter output $c(t)$ has its highest peak at the origin $t = 0$ without loss of generality, that is, $c_{\max} = c(0)$. Under this assumption, SNR in Eq. (4) is represented by

$$\begin{aligned}\mathrm{SNR} &= \frac{|E[c(0)]|^2}{\mathrm{Var}[c(0)]} \\ &= \frac{|\int H(f)X(f)df|^2}{\int |H(f)|^2 P_v(f)df}\end{aligned} \tag{5}$$

where $X(f)$ and $H(f)$ are Fourier transforms of $x(t)$ and $h(t)$, respectively. For given $X(f)$ and $P_v(f)$, SNR is maximized when we choose the linear filter $H(f)$ as follows:

$$H(f) = \alpha \frac{X^*(f)}{P_v(f)} \tag{6}$$

where α is any complex constant. Especially, when the input noise $v(t)$ is white noise, its PSD $P_v(f)$ is constant with respect to the frequency f, which yields

$$H(f) = \alpha X^*(f) \tag{7}$$

which is known as the *matched filter* since $H(f)$ is propotional to $X^*(f)$ or equivalently $h(t)$ is propotional to $x(-t)$.

C. Peak-to-Correlation Energy (PCE)

Peak-to-Correlation Energy (PCE) is a tractable correlation performance measure of the peak sharpness defined as

$$\begin{aligned}\mathrm{PCE} &= \frac{|c(0)|^2}{\int |c(t)|^2 dt} \\ &= \frac{|\int X(f)H(f)df|^2}{\int |X(f)H(f)|^2 df}\end{aligned} \tag{8}$$

Since maximal PCE is obtained when the correlation output is a delta function, the filter that maximizes PCE is

$$H(f) = \frac{1}{X(f)} \tag{9}$$

which is known as the *inverse filter*.

III. Phase-Only Correlation (POC) Functions

This section gives preliminaries about the POC filters. From this section, we deal with discrete-time signals.

A. Definition

Consider complex discrete-time signals $x(n)$ and $y(n)$ of length N. The discrete Fourier transforms of $x(n)$ and $y(n)$ are given by $X(k) = |X(k)|e^{j\theta_k}$ and $Y(k) = |Y(k)|e^{j\phi_k}$, respectively, where θ_k and ϕ_k are phase spectra of $x(n)$ and $y(n)$, respectively. The POC function $r(m)$ between two signals $x(n)$ and $y(n)$ is defined by the inverse discrete Fourier transform of normalized cross-power spectrum between two signals $x(n)$ and $y(n)$ as follows:

$$r(m) = \mathrm{IDFT}\left[\frac{X(k)Y^*(k)}{|X(k)Y^*(k)|}\right] = \frac{1}{N}\sum_{k=0}^{N-1} e^{j\alpha_k} W_N^{-mk} \quad (m = 0, 1, \cdots, N-1) \tag{10}$$

where $W_N = \exp(-j2\pi/N)$ is the twiddle factor, and $\alpha_k = \theta_k - \phi_k$ are *phase-spectrum differences*.

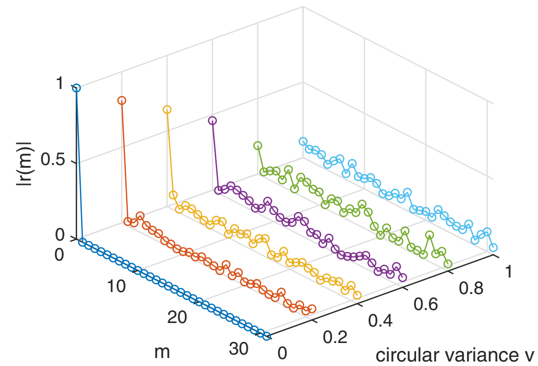

Fig. 2. POC functions $r(m)$ for various circular variances v of phase spectrum differences.

B. Properties

If two input signals are completely equal, phase spectra of two signals are equal, that is, $\alpha_k = \theta_k - \phi_k = 0$. In this case, the POC function $r(m)$ is the delta function $\delta(m)$. This property has been exploited in many matching techniques. However, in practical signal processing scene, it is quite unrealistic that the two input signals are equal. In most practical case, input signals are corrupted by noise, which causes corrupted phase spectra.

Figure 2 shows simple numerical examples of the POC functions for stochastic phase-spectrum differences. We set length of signals to be $N = 32$. We assume that phase-spectrum differences α_k's follow wrapped Gaussian distribution [10] with mean direction 0 and circular variance v, and calculate POC functions in Eq. (10) for circular variance $v = 0, 0.2, 0.4, 0.6, 0.8, 1$. We can observe that $|r(0)|$ decreases as the circular variance v increases. On the other hand, $|r(m \neq 0)|$ tend to increase as the circular variance v increases. In order to clarify the statistical properties of POC functions, we have to give some theoretical evidence for these experimental results of the POC functions with stochastic phase-spectrum differences. Behavior of peak value $|r(0)|$ can be described by expectation of the POC function $r(m)$. On the other hand, energy in sidelobe $|r(m \neq 0)|$ can be described by variance of the POC function $r(m)$.

IV. Statistical Analysis of the POC Functions Based on Directional Statistics

We proposed statistical analysis of phase-only correlation functions with stochastic phase-spectrum differences based on

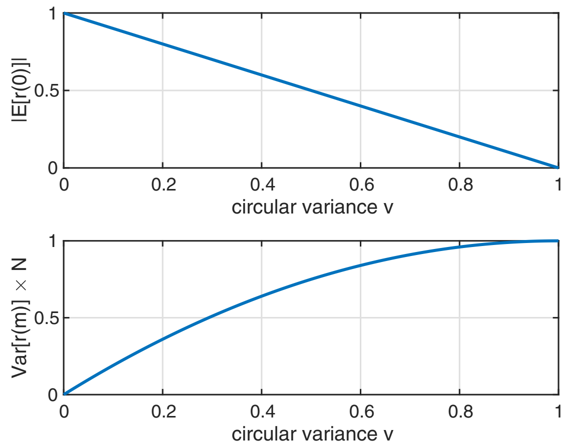

Fig. 3. Expectation $|E[r(0)]|$ and variance Var$[r(m)]$ of POC function $r(m)$ versus circular variance v.

directional statistics [9].

We derived the expectation and variance of the POC functions in terms of circular variance of phase-spectrum differences as follows:

Theorem 1: For i.i.d. stochastic phase-spectrum differences α_k's, the expectation and variance of POC functions $r(m)$ $(m = 0, 1, \cdots, N-1)$ are given by

$$\begin{aligned} E[r(m)] &= A\delta(m) \\ &= (1-v)e^{j\mu}\delta(m) \qquad (11) \\ Var[r(m)] &= \frac{1}{N}(1-|A|^2) \\ &= \frac{1}{N}\left(1-(1-v)^2\right) \qquad (12) \end{aligned}$$

where A, μ and v are the first-order trigonometric moment, mean direction and circular variance of phase-spectrum differences α_k's, respectively.

From Eqs. (11) and (12), the expectation $|E[r(0)]|$ and variance Var$[r(m)]$ versus circular variance v can be shown as Fig. 3. As circular variance v increases from 0 to 1, expectation $|E[r(0)]|$ monotonically decreases from 1 to 0, and variance Var$[r(m)]$ monotonically increases from 0 to 1. Furthermore, Eqs. (11) and (12) show that the expectation $E[r(m)]$ does not depend on signal length N, while the variance Var$[r(m)]$ is in inversely proportion to signal length N.

V. Correlation Performance Measures for POC Functions Based on Directional Statistics

In order to evaluate correlation performance of the POC functions, we newly derive the general expressions for correlation performance measures of the POC functions. In this section, the POC function $r(m)$ is assumed to have its peak at the origin $m = 0$, without loss of generality.

A. Signal-to-Noise Ratio (SNR)

Following the general definition of SNR, as correlation performance measures [5], we have defined SNR of the POC function by

$$\mathrm{SNR} = \frac{|E[r(0)]|^2}{\mathrm{Var}[r(0)]}. \qquad (13)$$

Substituting Eqs. (11) and (12) into Eq. (13), we have general expressions for SNR of the POC functions as follows:

$$\mathrm{SNR} = N\frac{|A|^2}{1-|A|^2}. \quad (0 \le |A| \le 1) \qquad (14)$$

On the other hand, based on directional statistics, we have derived general expressions for the expectation and variance of the POC function $r(m)$ in terms of circular variance v. From Eqs. (11) and (12), SNR in Eq. (14) can be expressed by simple fractional functions of circular variance v as follows:

$$\mathrm{SNR} = N\frac{(1-v)^2}{1-(1-v)^2}. \quad (0 \le v \le 1) \qquad (15)$$

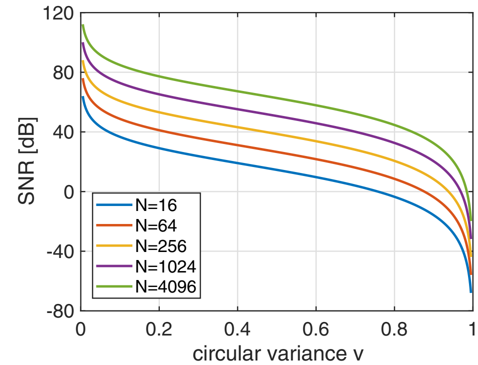

Fig. 4. SNR of the POC function $r(m)$ versus circular variance v.

Figure 4 shows SNR of the POC function $r(m)$ versus circular variance v. We can show from Eq. (15) and Fig. 4 that SNR monotonically decreases from $+\infty$ to 0 as circular variance v increases from 0 to 1. Furthermore, we can show that SNR is in proportion to signal length N.

B. Peak-to-Correlation Energy (PCE)

Following the general definition of PCE, as correlation performance measures [5], we have defined PCE of the POC function $r(m)$ by

$$\mathrm{PCE} = \frac{|r(0)|^2}{\sum_{m=0}^{N-1}|r(m)|^2}. \qquad (16)$$

It has been known that POC functions $r(m)$ always satisfy

$$\sum_{m=0}^{N-1}|r(m)|^2 = 1 \qquad (17)$$

which can be proved by Parseval's theorem. Therefore, PCE of the POC function $r(m)$ is simply given by

$$\mathrm{PCE} = |r(0)|^2. \qquad (18)$$

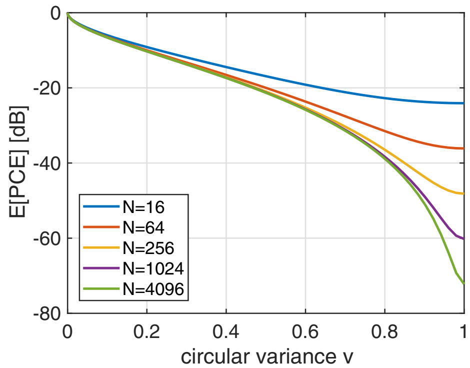

Fig. 5. Expectation of PCE of the POC function $r(m)$ versus circular variance v.

We can show from Eq. (18) that peak sharpness of the POC functions can be simply evaluated from only the squared peak value.

We next derive the expectation of PCE. Substituting Eq. (10) into Eq. (18), PCE of the POC function $r(m)$ can be expressed by

$$\text{PCE} = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} e^{j\alpha_k} e^{-j\alpha_l}. \tag{19}$$

By taking the expectation of Eq. (19), the expectation of PCE is given by

$$\begin{aligned} E[\text{PCE}] &= E\left[\frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} e^{j\alpha_k} e^{-j\alpha_l} \right] \\ &= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} E[e^{j\alpha_k} e^{-j\alpha_l}]. \end{aligned} \tag{20}$$

In the right side of Eq. (20), $E[e^{j\alpha_k} e^{-j\alpha_l}]$ is given by

$$E[e^{j\alpha_k} e^{-j\alpha_l}] = \begin{cases} 1 & (k = l) \\ |A|^2 & (k \neq l) \end{cases} \tag{21}$$

Substituting Eq. (21) into Eq. (20), we have general expressions for the expectation of PCE of the POC functions as follows:

$$E[\text{PCE}] = \frac{1}{N} + \left(1 - \frac{1}{N}\right) |A|^2. \quad (0 \leq |A| \leq 1) \tag{22}$$

On the other hand, by using the relationship $v = 1 - |A|$, $E[\text{PCE}]$ can be expressed in terms of the circular variance v as follows:

$$E[\text{PCE}] = \frac{1}{N} + \left(1 - \frac{1}{N}\right) (1 - v)^2. \quad (0 \leq v \leq 1) \tag{23}$$

Figure 5 shows the expectation of PCE of the POC function $r(m)$ versus circular variance v. We can show from Eq. (23) and Fig. 5 that $E[\text{PCE}]$ monotonically decreases from 1 to $1/N$ as circular variance v increases from 0 to 1. For large N, $E[\text{PCE}]$ can be approximated by $E[\text{PCE}] \approx (1 - v)^2$. Furthermore, it should be noted that $1/N \leq E[\text{PCE}] \leq 1$, which gives the minimum and maximum values of $E[\text{PCE}]$.

VI. Conclusions

In this paper, we proposed performance evaluation of the POC functions using SNR and PCE from the viewpoint of correlation filters. We derived the general expressions of SNR and PCE of the POC functions as correlation performance measures. SNR is expressed by simple fractional function of circular variance. PCE is simply given by squared peak value of the POC function, and its expectation is expressed in terms of circular variance.

These correlation performance measures are quite important in signal matching techniques based on POC functions, since they would give us useful criteria for discriminant between peak and sidelobe of the POC functions. In order to theoretically determine the optimal threshold value, we have to clarify the probability distributions of the POC functions based on directional statistics as our future works.

References

[1] C. D. Kuglin and D. C. Hines, "The phase correlation image alignment method," in *Proc. Int. Conf. Cybernetics and Society*, 1975, pp. 163–165.

[2] H. Foroosh, J. Zerubia, and M. Berthod, "Extension of phase correlation to subpixel registration," *IEEE Trans. Image Process.*, vol. 11, no. 3, pp. 188–200, Mar. 2002.

[3] L. Chen and K. H. Yap, "An effective technique for subpixel image registration under noisy condition," *IEEE Trans. Syst. Man. Cybern. A, Syst. Humans*, vol. 38, no. 4, pp. 881–887, July 2008.

[4] J. L. Horner and P. D. Gianino, "Pattern recognition with binary phase-only filters," *Applied Optics*, vol. 24, pp. 609–611, 1985.

[5] B. V. K. V. Kumar, A. Mahalanobis, and R. D. Juday, *Correlation Pattern Recognition*. Cambrige University Press, 2005.

[6] L. Hill and T. Vlachos, "Motion measurement using shape adaptive phase correlation," *IEE Electronics Letters*, vol. 37, no. 25, pp. 1512–1513, 2001.

[7] J. Ren, J. Jiang, and T. Vlachos, "High-accuracy sub-pixel motion estimation from noisy images in fourier domain," *IEEE Trans. Image Process.*, vol. 19, no. 5, pp. 1379–1384, May 2010.

[8] S. Yamaki, J. Odagiri, M. Abe, and M. Kawamata, "Effects of stochastic phase spectrum differences on phase-only correlation functions — Part I: Statistically constant phase spectrum differences for frequency indices —," in *Proc. IEEE Int. Conf. Network Infrastructure and Digital Content*, Beijing, China, Sept. 2012, pp. 360–364.

[9] S. Yamaki, M. Abe, and M. Kawamata, "Statistical analysis of phase-only correlation functions based on directional statistics," *IEICE Trans. Fundam. Electron., Commun., Comput., Sci.*, vol. E97-A, no. 12, pp. 2601–2610, Dec. 2014.

[10] ——, "Statistical analysis of phase-only correlation functions with phase-spectrum differences following wrapped distributions," *IEICE Trans. Fundam. Electron., Commun., Comput., Sci.*, vol. E99-A, no. 10, pp. 1790–1798, Oct. 2016.

[11] N. I. Fisher, *Statistical Analysis of Circular Data*. Cambridge University Press, 1993.

[12] K. V. Mardia and P. E. Jupp, *Directional Statistics*. John Wiley & Sons Ltd, 2000.