Proceedings of 2009 APSIPA Annual Summit and Conference, Sapporo, Japan, October 4-7, 2009

Digital Image Interpolation Method Using Higher-Order Hermite Interpolating Polynomials with Compact Finite-Difference

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***Abstract*—Image interpolation can be performed by a convolution operation using the neighboring image values. To achieve accurate image interpolation, some of the conventional methods use basis function with large support, and therefore their implementation may have a large computational cost. Interpolation by the Hermite interpolating polynomials can be performed using image values and their derivatives. This makes it possible to realize the high-order interpolation with small support. In this study, we show that, by introducing the basis function of derivatives, interpolation methods by the higher-order Hermite interpolating polynomials can be expressed as a convolution form similarly to the conventional methods. Thus, the higher-order interpolation with small support is obtained as well. Moreover, high accuracy is achieved by using the compact FDs as the calculation method of derivatives. As a result, the efficiency of this method is confirmed, by comparing it with the conventional methods which have the same support.**

I. Introduction

Image interpolation is an important technique to define a spatially continuous image from a set of discrete values. It is a fundamental method of many digital image processing operations, such as translation, scaling and rotation [1]. For these practical reasons, its accuracy is a significant concern. Commonly, the interpolated values at arbitrary locations are estimated by a convolution of the values at the surrounding points. However, to achieve high accuracy in image interpolation, the conventional methods use basis functions with large support, and therefore their implementation is quite complicated [1].

Interpolation function using Hermite interpolating polynomials can be expressed by image values and their derivatives. This makes it possible to realize the interpolation using the higher-order polynomials with small support. Consequently, the Hermite interpolation requires less support points than the conventional methods using only image values. For example, interpolation by the heptic (7th order) Hermite interpolating polynomials needs to use just four support points, while the conventional methods of the same order use eight points. This motivates the use of the Hermite interpolating polynomials for the accurate image interpolation.

In this study, we show the interpolation method by the higher-order (quintic, heptic) Hermite interpolating polynomials, which can be expressed as a convolution form, by introducing the basis function of derivatives. Moreover, we propose utilization of the compact finite difference (FD) in order to improve the calculation methods of the derivatives.

II. Conventional Interpolaion

We can express an interpolated value $F(x)$ at some (perhaps non-integer) coordinate x as the convolution form of coefficients c_k and basis function $h(x)$ at integer coordinate k [1]

$$F(x) = \sum_{k=-\infty}^{\infty} c_k h(x - x_k). \tag{1}$$

A. Linear Interpolation

Linear interpolation is widely used in image interpolation because of its implementation simplicity. However, this technique attenuates high frequency components of the image. Let f_k be the image values. In linear interpolation, we use f_k for c_k. The basis function of linear interpolation is given by Eq. (2). (See Fig. 1) The derivative of the interpolated function is not continuous at the integer points

$$h(x) = \begin{cases} -|x| + 1, & 0 \leq |x| < 1 \\ 0, & 1 \leq |x| \end{cases} . \tag{2}$$

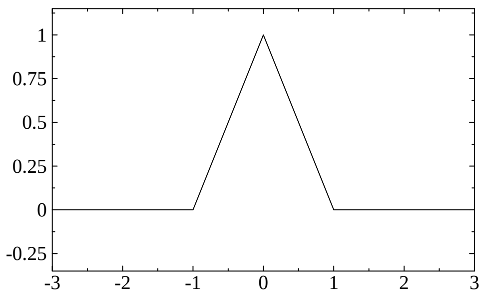


Fig. 1. Basis function of linear interpolation.

B. Cubic Convolution

Because cubic convolution performs better than linear interpolation, it is also widely used, despite more complexity. Its function is defined as piecewise cubic polynomials, which depend on a parameter a [2]. Its general expression is given by Eq. (3), and is depicted in Fig. 2

$$h(x) = \begin{cases} (a+2)|x|^3 - (a+3)|x|^2 + 1, & 0 \le |x| < 1 \\ a|x|^3 - 5a|x|^2 + 8a|x| - 4a, & 1 \le |x| < 2 \\ 0, & 2 \le |x| \end{cases} . \tag{3}$$

In this interpolation, as well as in the linear interpolation, image values f_k are used for c_k. Although this method controls the characteristics of the interpolation by varying a, it has lower accuracy than the cubic B-spline interpolation method, described in the following section.

C. Cubic B-spline

The basis function of the cubic B-Spline interpolation is expressed as

$$h(x) = \begin{cases} \frac{|x|^3}{2} - |x|^2 + \frac{2}{3}, & 0 \le |x| < 1 \\ -\frac{|x|^3}{6} + |x|^2 - 2|x| + \frac{4}{3}, & 1 \le |x| < 2 \\ 0, & 2 \le |x| \end{cases} . \tag{4}$$

Its form is shown in Fig. 3. In this method, we don't utilize f_k for c_k, i.e., $f_k \ne c_k$. That is, we need the following procedure:

i) The determination of coefficients c_k from image values f_k;
ii) The determination of required value $F(x)$ from the convolution of c_k and its basis function.

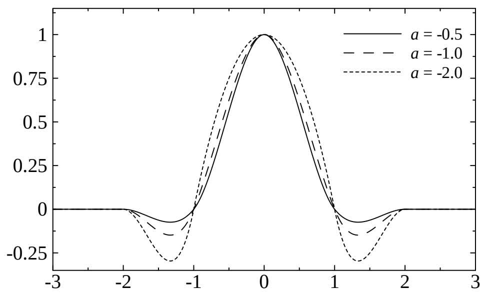


Fig. 2. Basis function of cubic convolution.

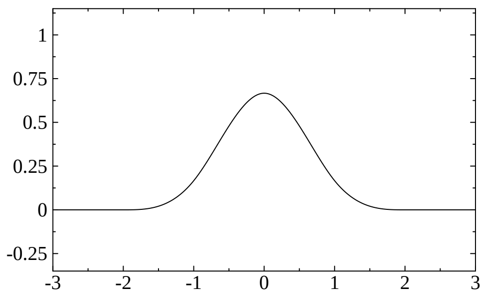


Fig. 3. Basis function of cubic B-spline.

In the cubic B-spline interpolation, the coefficients c_k are determined by using recursive filter [3], [4]. However, for higher-order splines with large support, the filter must be used several times, which may have a larger computational cost.

III. Interpolation by Hermite Interpolating Polynomials

In this section, we describe the proposed interpolation by the Hermite interpolating polynomials.

A. Basis Function

In interpolation using the Hermite interpolating polynomials, the interpolated value $F(x)$ can be expressed as a unique form of convolution of image values f_k and derivatives f'_k:

$$F(x) = \sum_{k=-\infty}^{\infty} f(x_k)h(x - x_k) + \sum_{k=-\infty}^{\infty} f'(x_k)h_d(x - x_k). \tag{5}$$

Its overall benefit is to allow us to use the higher-order basis functions which require small support and also have high accuracy. For example, the basis functions $h(x)$ and $h_d(x)$ for the cubic, the quintic and the heptic Hermite interpolation are given by Eqs. (6) to (11), respectively, and are depicted in Figs. 4 to 6

$$h(x) = \begin{cases} 2|x|^3 - 3|x|^2 + 1, & 0 \le |x| < 1 \\ 0, & 1 \le |x| \end{cases} , \tag{6}$$

$$h_d(x) = \begin{cases} x^3 - 2x^2 + x, & 0 \le x < 1 \\ x^3 + 2x^2 + x, & -1 < x < 0 \\ 0, & 1 \le |x| \end{cases} . \tag{7}$$

$$h(x) = \begin{cases} -\frac{3}{4}|x|^5 + \frac{7}{4}|x|^4 + \frac{3}{4}|x|^3 - \frac{11}{4}|x|^2 + 1, & 0 \le |x| < 1 \\ \frac{(|x|-1)^2}{12}(-|x|^3 + 7|x|^2 - 16|x| + 12), & 1 \le |x| < 2 \\ 0, & 2 \le |x| \end{cases} , \tag{8}$$

$$h_d(x) = \begin{cases} -\frac{x^5}{2} + \frac{3}{2}x^4 - \frac{x^3}{2} - \frac{3}{2}x^2 + x, & 0 \le x < 1 \\ -\frac{x^5}{2} - \frac{3}{2}x^4 - \frac{x^3}{2} + \frac{3}{2}x^2 + x, & -1 < x < 0 \\ 0, & 1 \le |x| \end{cases} . \tag{9}$$

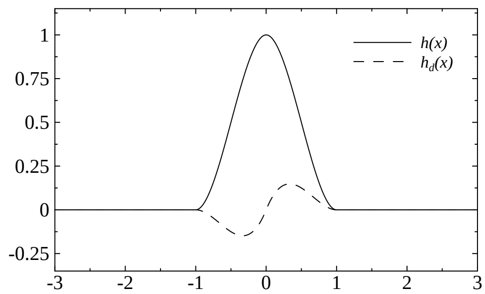


Fig. 4. Basis function of cubic Hermite.

$$h(x)=\begin{cases}\frac{|x|^7}{4}-\frac{3}{4}|x|^6-\frac{|x|^5}{2}+\frac{5}{2}|x|^4+\frac{|x|^3}{4}-\frac{11}{4}|x|^2+1 & 0\le|x|<1\\ (|x|-1)^2(\frac{11}{108}|x|^5-\frac{107}{108}|x|^4+\frac{377}{108}|x|^3-\frac{61}{12}|x|^2+2|x|+1) & 1\le|x|<2\\ 0 & 2\le|x|\end{cases}, \tag{10}$$

$$h_d(x)=\begin{cases}\frac{x^7}{4}-x^6+\frac{x^5}{2}+2x^4-\frac{7}{4}x^3-x^2+x & 0\le x<1\\ \frac{(x-1)^2}{36}(x^5-10x^4+37x^3-60x^2+36x) & 1\le x<2\\ \frac{x^7}{4}+x^6+\frac{x^5}{2}-2x^4-\frac{7}{4}x^3+x^2+x & -1<x<0\\ \frac{(x+1)^2}{36}(x^5+10x^4+37x^3+60x^2+36x) & -2<x\le-1\\ 0 & 2\le|x|\end{cases}. \tag{11}$$

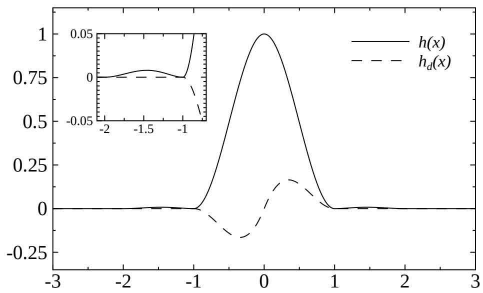


Fig. 5. Basis function of quintic Hermite.

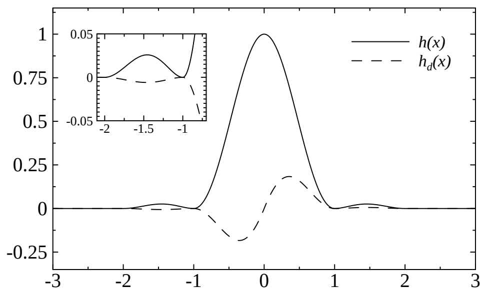


Fig. 6. Basis function of heptic Hermite.

B. Calculation of Derivatives

In interpolation by the Hermite interpolating polynomials, it is important how to calculate the value of derivatives f'_k, because derivatives sufficiently influence the accuracy of interpolation. The approximation of the first derivatives by FDs is derived from Taylor series. The second and fourth-order central FDs at a point i are given by Eqs. (12) and (13), respectively

$$f'_i=\frac{1}{2}(f_{i+1}-f_{i-1}), \tag{12}$$

$$f'_i=\frac{-f_{i+2}+8f_{i+1}-8f_{i-1}+f_{i-2}}{12}, \tag{13}$$

where f_i and f'_i are the values and their derivatives at i, respectively.

On the other hand, the compact FDs at a point i are obtained by relation equations of values (e.g. f_{i-2}, f_{i-1}, f_{i+1}, f_{i+2}) and their derivatives (e.g. f'_{i-2}, f'_{i-1}, f'_{i+1}, f'_{i+2}). For example, relation equations for fourth and sixth-order compact FDs are given by Eqs. (14) and (15), respectively

$$\frac{1}{4}f'_{i+1}+f'_i+\frac{1}{4}f'_{i-1}=\frac{3}{4}(f_{i+1}-f_{i-1}), \tag{14}$$

$$\frac{1}{3}f'_{i+1}+f'_i+\frac{1}{3}f'_{i-1}=\frac{f_{i+2}+28f_{i+1}-28f_{i-1}-f_{i-2}}{36}. \tag{15}$$

Compared with the central FDs approximation of the same order, the compact FDs provide higher accuracy [5], [6]. These two compact FDs can be calculated in the same way as determining the cubic B-spline coefficients c_k. Thus, by using these two compact FDs, we can obtain higher-order expression of the Hermite interpolation which has almost as large computational cost as the cubic B-spline has.

IV. Experiments

To evaluate the interpolation results, the following strategy was adopted; we apply a succession to the image by rotation done r times at degree θ, where $r=36$ and $\theta=2\pi/36=10°$. Thus, we compare the initial image with the final one, and show the SNR values. Here, to avoid the boundary effects, we made a comparison only on the central portion of the final image.

Figure 7 (a) and (b) show 256 $\times$ 256 pixels test images. Figure 8 (a) and (b) show the central portion of these images. Then, Fig. 9 shows the results of rotation experiment when using each interpolation method. Linear interpolation (Fig.9 (a)) performs worst, as compared with other methods. The reason for this is the dissipation of high frequency components, which causes blurring in the image. Cubic convolution with the parameter a = -0.5 (Fig.9 (b)) and the Hermite interpolation using central FDs (Fig.9 (d)) provide poor visual performance; a part of the stripe pattern of the clothes seems to be blurred. On the other hand, cubic B-spline (Fig.9 (c)) and the Hermite interpolation using compact FDs (Fig.9 (e) and (f)) result in less blurring. However, a careful comparison shows that the stripe pattern is more clear in the heptic Hermite interpolation using sixth-order compact FD than that in the cubic B-spline. Here, we must notice that both cubic B-spline and heptic Hermie interpolation have the same support, and require almost the same computational cost due to determing the coefficients c_k or derivatives f'_i.

(a) (b)

Fig. 7. Test image. (a)Barbara. (b)Lena.

(a) (b)

Fig. 8. Central portion of test image. (a)Barbara. (b)Lena.

Table I presents the numerical results of these experiments, along with some additional methods. In particular, we also provide the results for the standard Lena test image of Fig. 8. The SNR for evaluation is defined as

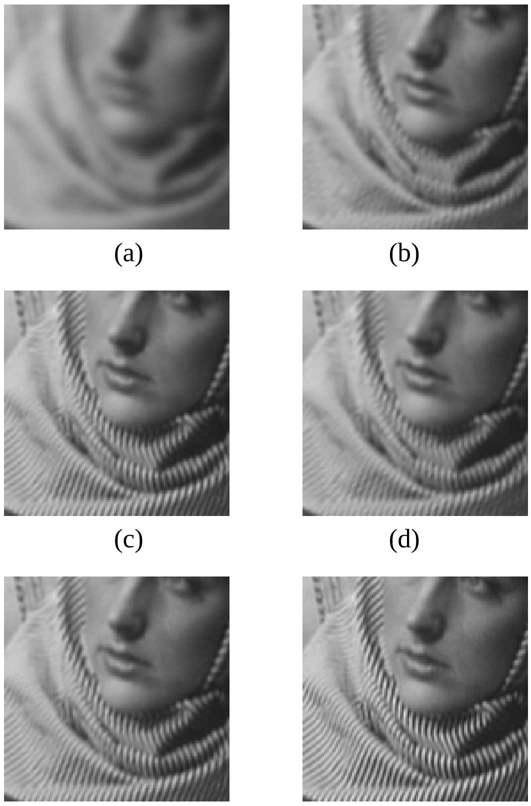
$$\mathrm{SNR} = 10\log_{10}\frac{\sum_{k=1}^{N} f(x_k)^2}{\sum_{k=1}^{N}\left(f(x_k)-g(x_k)\right)^2}, \qquad (16)$$

where $f(x_k)$ is the original data, and $g(x_k)$ is the result of the rotation.

These results illustrate the efficiency of the image interpolation by the Hermite interpolating polynomial, as compared with other methods which have the same support.

V. Conclusions

In this study, we have expressed the interpolation by the higher-order (quintic, heptic) Hermite interpolating polynomials as a convolution form, by introducing the basis function of derivatives. Thus, the higher-order expressions with small support are obtained as well. Moreover, high accuracy has been achieved by using the compact FDs as the calculation method of the derivatives. The performance of the proposed method has been verified by comparison with some of the conventional methods for the same support. The comparison shows that the proposed method provides higher accuracy than the conventional methods.



(a) (b) (c) (d) (e) (f)

Fig. 9. Roation of Barbara. (a)Linear. (b)Cubic convolution ($a = -0.5$). (c)Cubic B-spline. (d)Cubic Hermite (4^{th}-order central FD). (e)Cubic Hermite (4^{th}-order compact FD). (f)Heptic Hermite (6^{th}-order compact FD).

TABLE I
Results in Numerical Form

Basis function	**Barbara[dB]**	**Lena[dB]**
Linear	15.38	17.80
Cubic convolution ($a = -0.5$)	17.07	22.73
Cubic B-spline	19.96	26.13
Cubic Hermite (2^{nd}-order central FD)	17.07	22.73
Cubic Hermite (4^{th}-order central FD)	18.29	24.60
Cubic Hermite (4^{th}-order compact FD)	19.96	26.13
Cubic Hermite (6^{th}-order compact FD)	21.39	27.27
Quintic Hermite (2^{nd}-order central FD)	17.07	22.70
Quintic Hermite (4^{th}-order central FD)	18.53	24.87
Quintic Hermite (4^{th}-order compact FD)	20.89	26.92
Quintic Hermite (6^{th}-order compact FD)	23.03	28.67
Heptic Hermite (2^{nd}-order central FD)	16.93	22.44
Heptic Hermite (4^{th}-order central FD)	18.39	24.75
Heptic Hermite (4^{th}-order compact FD)	20.99	27.00
Heptic Hermite (6^{th}-order compact FD)	23.85	29.06

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