Proceedings of 2009 APSIPA Annual Summit and Conference, Sapporo, Japan, October 4-7, 2009

EQUAL GAIN BEAMFORMING IN RAYLEIGH FADING CHANNELS

Shang-Ho Tsai

Department of Electrical Engineering
National Chiao Tung University
Hsinchu, Taiwan 300
Email: shanghot@mail.nctu.edu.tw

ABSTRACT

Precoding techniques with limited feedback information can achieve satisfactory performance while the amount of feedback information is kept small. In this paper, we analyze the theoretical performance of equal gain precoder and find that its performance is at most 1.049 dB worse than the optimal precoder no matter how the number of transmit antennas increases. Moreover, we found that the equal gain precoder in general can achieve comparable performance with the Grassmannian precoder in the same moderate feedback bits. Hence, it has several advantages compared with the Grassmannian precoder that we will explain later in the paper.

Index Terms— MIMO, precoding, beamforming, equal gain precoding, Grassmannian beamforming

1. INTRODUCTION

MIMO techniques are widely used in current wireless communication standards such as IEEE 802.11n and IEEE 802.16. Among the MIMO skills, precoding/beamforming can provide full diversity order and additional precoding gain. Such nice properties can greatly improve system performance.

The amount of feedback information plays an important role in precoder designs. If complete channel formation is known to the transmitter, the optimal performance can be achieved [3]. However, such precoding scheme leads to a large amount of feedback information. To overcome this, research has been directed to the precoding schemes with limited feedback recently. In [3], equal gain precoder with different combining methods were shown to achieve full diversity order of MIMO channels. In [4], Grassmannian beamforming/precoding was proposed. This precoder was shown to have good performance in practical communication systems. The Grassmannian precoder first needs to construct a codebook using Grassmannian packing theory. Then it determines the best codeword using exhaustive search.

In this paper, we analyze the theoretical performance of the equal gain precoder in MISO channel environments and found several interesting results as follows:

First, the BEP (bit error probability) performance gap between the equal gain precoder and the optimal precoder varies from 0.5 dB to 1.049 dB as the number of transmit antennas grows from 2 to infinity. This is a somewhat surprising result since the equal gain precoder only needs the channel phase information while its performance is worse than the optimal precoder, which needs both magnitude and phase information, by at most 1.049 dB no matter how the number of transmit antennas increases. As the number of transmit antennas increases, the required magnitude and phase information also increases. In this case, using equal gain precoder can greatly reduce the feedback overhead but at the same time, it also maintains satisfactory performance.

Second, we found from simulation results that the equal gain precoder can achieve comparable performance with the Grassmannian precoder in the same moderate number of feedback bits. This result shows several advantages of the equal gain precoder. 1) The computational complexity of the equal gain precoder is lower than the Grassmannian precoder, since in MISO channel environments the optimal solution for the equal gain precoder can be obtained directly from the channel information [3]. Hence there is no need to perform exhaustive search to determine the codeword as Grassmannian does. 2) In the equal gain precoder, when the feedback bits are less than two per transmit antenna, there is no need to perform multiplications in the transmitter side since the codeword coefficients in this case are ± 1 or $\pm j$. On the other hand, the Grassmannian precoder needs to perform complex multiplications in the transmit side. 3) It may be difficult for the Grassmannian precoder to construct the codebook for a large number of transmit antennas [4] due to the large number of combinations. On the other hand, since the equal gain precoder has close form solution in the MISO case, it can be easily extended to arbitrary number of transmit antennas.

Notations: Boldfaced lowercases and **Boldfaced** uppercases denote vectors and matrices, respectively. $\mathbb{E}\{x\}$ denotes the expectation of random variable x. $\mathbf{A}^*$ and $\mathbf{A}^t$ denote the conjugate and transpose of $\mathbf{A}$, respectively. $\mathbf{A}^\dagger$ is the conjugate-transpose of $\mathbf{A}$. $\Re\{x\}$ denotes the real part of variable x. σ_x^2 is the variance of random variable x.

2. SYSTEM MODEL

The block diagram of the equal gain precoder is shown in Fig. 1. Let the number of transmit antennas be N_t. At the first stage, one transmit symbol (can be complex such as QPSK and M-QAM) is sent to N_t branches for precoding. Each symbol in different branch is multiplied by a different phase rotation, *i.e.* $e^{j\theta_1}/\sqrt{N_t}, \cdots e^{j\theta_{N_t}}/\sqrt{N_t}$, where dividing $\sqrt{N_t}$ is to remain the same total transmit power. The phase information is feeded back from the receiver side according to the channel condition. In the analysis of this paper, perfect phase information is assumed to obtain the performance gap between the optimal and the equal gain precodes. However, simulation result will show performance comparison with limited feedback. After the precoding, the symbol vector, $\mathbf{s} = (s_1\ s_2\ \cdots s_{N_t})^t$, to be transmitted is given by

$$\mathbf{s} = \frac{1}{\sqrt{N_t}}\mathbf{p}x, \tag{1}$$

where $\mathbf{p}$ is a $N_t \times 1$ vector defined as

$$\mathbf{p} = \left(e^{j\theta_1}\ e^{j\theta_2}\ \cdots e^{j\theta_{N_t}}\right)^t. \tag{2}$$

Then, $\mathbf{s}$ is transmitted to the channel.

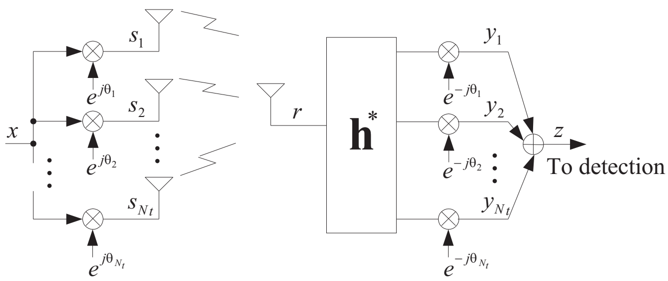

Fig. 1. A block diagram of the equal gain precoder.

At the receive side, the received symbol r is given by

$$r = \mathbf{h}^t\mathbf{s} + n, \tag{3}$$

where $\mathbf{h}$ is a $N_t \times 1$ channel vector consisting of channel coefficient given by

$$\mathbf{h} = (h_1\ h_2\ \cdots\ h_{N_t})^t, \tag{4}$$

and n is a noise scalar. The receive symbol r is first multiplied by the conjugate of the channel vector, *i.e.* $\mathbf{h}^*$. Then, it is multiplied by the inverse phase rotation and form a scalar z. The mathematical expression is given by

$$z = \mathbf{p}^\dagger\mathbf{h}^* r. \tag{5}$$

From Eqs. (1), (3) and (5), the relationship between x and z are given by

$$z = \frac{1}{\sqrt{N_t}}\underbrace{\mathbf{p}^\dagger\mathbf{h}^*\mathbf{h}^t\mathbf{p}}_{\gamma}x + \mathbf{p}^\dagger\mathbf{h}^* n, \tag{6}$$

where $\mathbf{p}^\dagger\mathbf{h}^* n$ is the noise after decoding, and γ is a gain effect (including diversity gain and precoding gain) due to the precoding. Note that it can be easily shown thatγ is real. Then the symbol z is ready for detection.

3. PERFORMANCE ANALYSIS

In this section, let us analyze the performance of the equal gain precoder. From (6), for a given channel realization (channel is deterministic), the instantaneous SNR of the equal gain precoder can be shown to be

$$SNR_{EG} = \frac{\gamma}{N_t}\frac{\sigma_x^2}{\sigma_n^2}. \tag{7}$$

From (7), for a given channel realization and σ_x^2/σ_n^2, the instantaneous SNR is determined by γ. It can be shown that γ is upper bounded as

$$\gamma \leq \sum_{i=1}^{N_t}|h_i|^2 + 2\Re\left\{\sum_{i=1}^{N_t}\sum_{j>i}^{N_t}|h_i^* h_j|\right\}. \tag{8}$$

To minimize the average bit error probability, we need to maximize the average SNR (see [1] and [4]). From (7) and (8), the average SNR can be upper bounded by

$$\begin{aligned} &\mathbb{E}_{\mathbf{h}}\left\{SNR_{EG}\right\} \\ \leq\ & \frac{1}{N_t}\frac{\sigma_x^2}{\sigma_n^2}\left(\sum_{i=1}^{N_t}\mathbb{E}_{\mathbf{h}}\left\{|h_i|^2\right\} + \sum_{i=1}^{N_t}\sum_{j\neq i}^{N_t}\mathbb{E}_{\mathbf{h}}\left\{|h_i^* h_j|\right\}\right) \\ =\ & \frac{\sigma_x^2}{\sigma_n^2}\left(\mathbb{E}_{\mathbf{h}}\left\{|h_i|^2\right\} + (N_t-1)\mathbb{E}_{\mathbf{h}}\left\{|h_i^* h_j|\right\}\right). \end{aligned} \tag{9}$$

From (9), if both $\mathbb{E}_{\mathbf{h}}\left\{|h_i|^2\right\}$ and $\mathbb{E}_{\mathbf{h}}\left\{|h_i^* h_j|\right\}$ are known, the average SNR for equal gain precoding is known. Without losing the generality, let us assume that both the real part and the imaginary part of h_i have unit variance. In this case, $\mathbb{E}_{\mathbf{h}}\left\{|h_i|^2\right\} = \sigma_h^2 = 2$. Next, let us see how to obtain $\mathbb{E}_{\mathbf{h}}\left\{|h_i^* h_j|\right\}$ in the following lemmas and theorem.

Lemma 1: Assume that h_i is complex Gaussian with zero mean and unit variance in both the real part and the imaginary part. The probability density function (PDF) of $|h_i^* h_j|$ is given by

$$f_h(x) = \frac{x}{2}\left(K_o(x) + K_2(x)\right) - K_1(x), \tag{10}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind.

Proof: For h_i being complex Gaussian with zero mean and unit variance in both the real part and the imaginary part, the cumulative distribution function (CDF) of $|h_i^* h_j|$ can be shown to be (see [6])

$$F_h(x) = 1 - xK_1(x). \tag{11}$$

According to [7], we have the following equality for the derivative of the modified Bessel function of the second kind $K_\nu(x)$:

$$\frac{\partial K_\nu(x)}{\partial x} = -\frac{1}{2}\left(K_{\nu-1}(x) + K_{\nu+1}(x)\right). \tag{12}$$

From (11) and (12), and using the fact that $f_h(x) = \frac{\partial F_h(x)}{\partial x}$, we can obtain the PDF in (10).

$\triangle\triangle\triangle$

Lemma 2: Assume that h_i is complex Gaussian with zero mean and unit variance in both the real part and the imaginary part. The mean of the random variable $|h_i^* h_j|$ can be calculated as

$$\mathbb{E}_{\mathbf{h}}\left\{|h_i^* h_j|\right\} = 1.5708. \tag{13}$$

Proof: According to [8], we have the following equality for the integration of the modified Bessel function of the second kind, $K_\nu(x)$:

$$\int_0^\infty x^{\alpha-1} K_\nu(x)dx = 2^{\alpha-2}\Gamma\left(\frac{\alpha-\nu}{2}\right)\Gamma\left(\frac{\alpha+\nu}{2}\right), \tag{14}$$

where $\Gamma(x)$ is the gamma function. From Lemma 1, the expectation value of $|h_i^* h_j|$ can be expressed as

$$\begin{aligned}\mathbb{E}_{\mathbf{h}}\left\{|h_i^* h_j|\right\} &= \int_{-\infty}^{\infty} x f_h(x)dx \\ &= \int_0^\infty \frac{x^2}{2}K_0(x)dx + \int_0^\infty \frac{x^2}{2}K_2(x)dx \\ &\quad - \int_0^\infty xK_1(x)dx, \end{aligned} \tag{15}$$

where we have used the fact that $K_\nu(x) = 0$, for $x < 0$. From (14), we can rewrite $\mathbb{E}_{\mathbf{h}}\left\{|h_i^* h_j|\right\}$ as

$$\Gamma(3/2)\Gamma(3/2) + \Gamma(1/2)\Gamma(5/2) - \Gamma(1/2)\Gamma(3/2) = 1.5708.$$

$\triangle\triangle\triangle$

Theorem: Assume that the channel coefficient h_i is complex Gaussian with zero mean. The average SNR gap of the optimal precoder (unlimited feedback) and the equal gain precoder (unlimited feedback) in a MISO channel is given by

$$\frac{\mathbb{E}_{\mathbf{h}}\left\{SNR_{OPT}\right\}}{\mathbb{E}_{\mathbf{h}}\left\{SNR_{EG}\right\}} = \frac{N_t}{0.7854N_t + 0.2416}. \tag{16}$$

Proof: For the optimal precoder [3], the precoding vector is the eigenvector corresponding to the maximum eigenvalue of $\mathbf{h}\mathbf{h}^\dagger$. In the MISO case, this optimal precoder can be chosen as $\mathbf{h}^\dagger$. Its average SNR can be shown to be

$$\mathbb{E}_{\mathbf{h}}\left\{SNR_{OPT}\right\} = N_t \frac{\sigma_x^2}{\sigma_n^2}\mathbb{E}_{\mathbf{h}}\left\{|h_i|^2\right\}. \tag{17}$$

To have a fair comparison, we also assume that the channel coefficients in the optimal precoder have unit variance in both the real and the imaginary parts, since we assume the same condition for the equal gain precoder in Lemma 1 and Lemma 2 as well. Thus, $\mathbb{E}_{\mathbf{h}}\left\{|h_i|^2\right\} = 2$. From (17), we have

$$\mathbb{E}_{\mathbf{h}}\left\{SNR_{OPT}\right\} = \frac{\sigma_x^2}{\sigma_n^2}2N_t. \tag{18}$$

For the equal gain precoder, from (9) and (13), we have

$$\begin{aligned}\mathbb{E}\left\{SNR_{EG}\right\} &\leq \frac{\sigma_x^2}{\sigma_n^2}\left(\mathbb{E}_{\mathbf{h}}\left\{|h_i|^2\right\} + (N_t-1)\mathbb{E}_{\mathbf{h}}\left\{|h_i^* h_j|\right\}\right) \\ &= \frac{\sigma_x^2}{\sigma_n^2}(2 + (N_t-1)1.5708). \end{aligned} \tag{19}$$

The equality holds when full phase information is known to the transmitter. From (18) and (19), we obtain the result in (16). Please note that since this is a fair comparison for the two precoders, there is no need to constrain the variance of h_i in the theorem.

$\triangle\triangle\triangle$

From the theorem, when $N_t \gg 1$, the ratio approximates $1/0.7854 = 1.2732$ = 1.049 (dB), which is a constant performance gap between the optimal precoder and the equal gain precoder. For small N_t, the ratio is smaller than 1.049 (dB). Taking $N_t = 2$ for instance, the ratio is $1.1035 = 0.4278$ (dB). This is an interesting result because it means that the performance loss due to the use of phase alone (without magnitude) is at most around 1 dB, despite the increase of the transmit antennas. Since the representation for both magnitude and phase require much more bits and feedback effort than that for phase alone, this theorem offers an important strategy that when feedback capacity is limited, we may use the equal gain precoding since its performance loss is at most around 1 dB compared to the optimal precoder. Let us see the following simulation examples to illustrate this point.

4. SIMULATION RESULT

The simulation is conducted using the following parameters: The channel coefficients are i.i.d. complex Gaussian random variables with zero mean and unit variance. One receive antenna was used. The modulation level is 16-QAM. The SNR in the simulation was defined as the ratio of average symbol power in the transmitter to noise power. For description convenience, we let b be the number of bits to represent the phase of each transmit antenna (except the first antenna) for the equal gain precoder [3]. Moreover, we let B be the number of total feedback bits for all antennas. Thus, we have the relationship that $B = (N_t - 1)b$ for the equal gain precoder.

Example 1: Comparison of the optimal and the equal gain precoders. Let us compare the performance between the optimal and the equal gain precoders. To see the best performance that these two precoders can achieve, we do not quantize the precoding vectors $\mathbf{p}$ in this example. Fig. 2 shows the bit error probability (BEP) performance of the optimal and the equal gain precoders without quantization. We observe that the optimal precoder outperforms the equal gain precoder around 0.5 dB when $N_t = 2$. When the number of transmit antennas increases, the performance gap increases. However, the increasing speed soon decreases. For instance, when $N_t = 8$, the gap is around 0.9 dB and when $N_t > 16$, the gap is around 1 dB. This result shows that the performance gap between the optimal and the equal gain precoders is around 1 dB, which corroborates the theoretical result in (16).

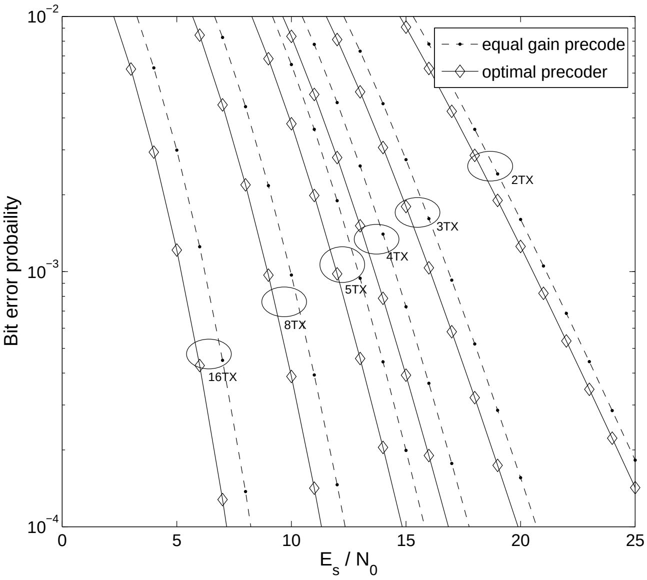

Fig. 2. Performance comparison of the optimal precoder and the equal gain precoder without quantization.

Example 2: Comparison of various precoders with quantization effect. In this example, we compare the performance of various precoders including the equal gain precoder, the Grassmannian precoder [3] and the antenna selection precoder [2]. The performance comparison is shown in Fig. 3. For fair comparison, total required bits are shown for all precoders. To evaluate the performance improvement due to precoding, we also include the 2×1 STBC performance as shown in the solid-square curve. From the figure, we see that the three precoders have the same diversity gain and hence their slopes are the same. However, the Grassmannian and the equal gain precoders can achieve a better performance than the antenna selection precoder. Moreover, we see in this simulation case that with the same required total bits B, the equal gain precoder can achieve comparable performance with the Grassmannian precoder (less than 0.2 dB performance gap).

5. CONCLUSION

We analyzed the theoretical performance of the equal gain precoder. We found that the performance gap between the optimal precoder and the equal gain precoder is only around 1 dB. Moreover, we showed that generally the equal gain precoder can achieve comparable performance while lower complexity when compared with the Grassmannian precoder.

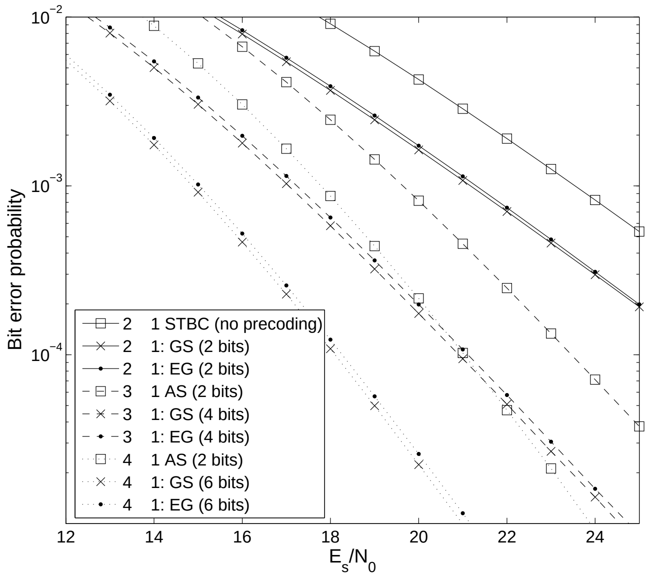

Fig. 3. Performance comparison for the space-time block code (STBC), equal gain (EG) precoder, Grassmannian (GS) precoder and antenna selection (AS) precoder.

6. REFERENCES

[1] P. A. Dighe, R. K. Mallink and S. S. Jamuar, "Analysis of transmit-receiver diversity in Rayleigh fading," *IEEE Trans. Commun.,* Apr. 2003.

[2] D. A. Gore and A. J. Paulraj, "MIMO antenna subset selection with space-time coding," *IEEE Trans. Signal Process.,* Oct. 2002.

[3] D. J. Love and R. W. Heath "Equal gain transmission in multiple-input wireless systems," *IEEE Trans. Commun.,* Jul. 2003.

[4] D. J. Love and R. W. Heath "Grassmannian beamforming for multiple-input multiple-output wireless systems," *IEEE Trans. Inform. Theory,* Oct. 2003.

[5] http://cobweb.ecn.purdue.edu/∼djlove/grass

[6] W. T. Wells, et. al. "The Distribution of the Product of Two Central or Non-Central Chi-Square Variates," *The Annals of Mathematical Statistics,* Sep. 1962.

[7] http://functions.wolfram.com/03.04.20.0006.01

[8] http://functions.wolfram.com/03.04.21.0116.01