Structural Sub-band Decomposition: A New Concept in Digital Signal Processing

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Outline

- **Signal and System Decomposition**
  – Polyphase decomposition
  – Structural subband decomposition
- **Subband Discrete Transforms**
  - Subband discrete Fourier transform
  - Subband discrete cosine trasform
  - Applications
- **Subband FIR Filter Design and Implementation**
- **Subband Adaptive Filtering**

Polyphase Decomposition

- In the M-band polyphase decomposition, a sequence $\{x[n]\}$ is expressed as a sum of M subsequences $\{x_i[n]\}$, $0 \le n \le M-1$, obtained by down-sampling $\{x[n]\}$ by a factor of M with i indicating the phase of the sub-sampling process

$$x_i[n] = x[Mn+i],$$

Polyphase Decomposition

- For example, for $M = 2$, for a causal sequence $\{x[n]\}$, the two sub-sequences are:

$$\{x_0[n]\} = \{x[0] \quad x[2] \quad x[4] \quad x[6] \quad \cdots\}$$

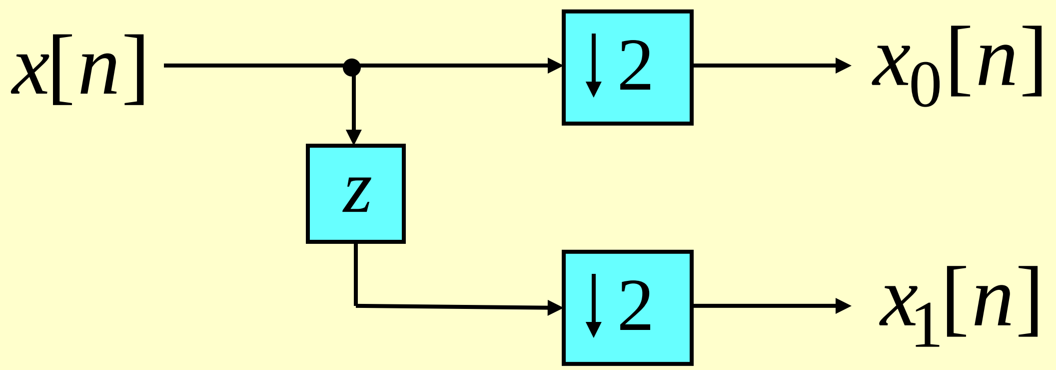
- Even samples of $\{x[n]\}$

$$\{x_0[n]\} = \{x[0] \quad x[2] \quad x[4] \quad x[6] \quad \cdots\}$$

- Odd samples of $\{x[n]\}$

Polyphase Decomposition

- Physical Interpretation – 2-Band Case



Polyphase Decomposition

- Likewise, for $M = 3$, for a causal sequence $\{x[n]\}$, the three sub-sequences are:

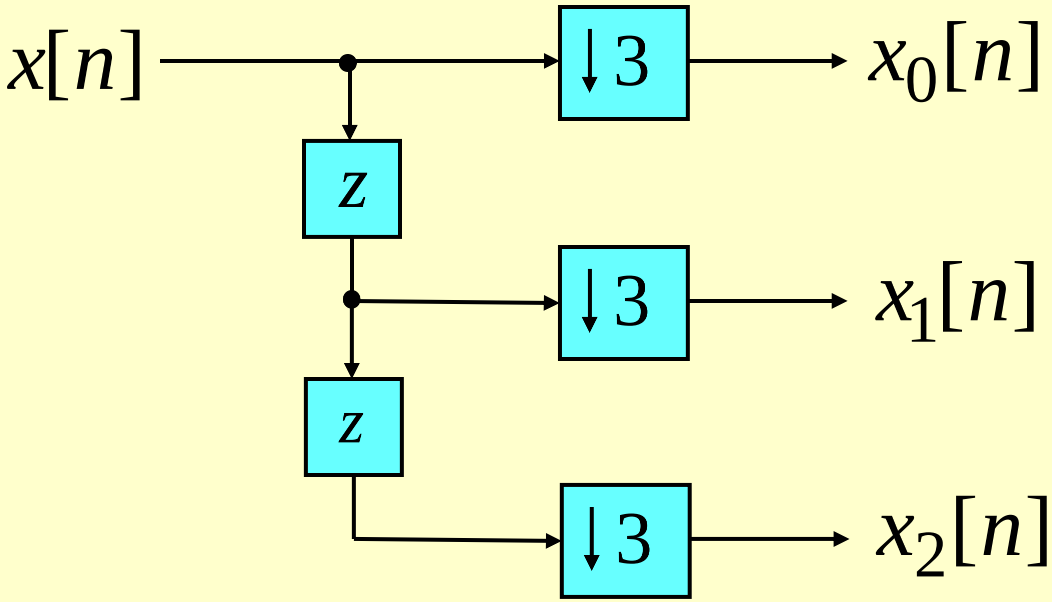
$$\{x_0[n]\} = \{x[0] \quad x[3] \quad x[6] \quad x[9] \quad \cdots\}$$

$$\{x_1[n]\} = \{x[1] \quad x[4] \quad x[7] \quad x[10] \quad \cdots\}$$

$$\{x_2[n]\} = \{x[2] \quad x[5] \quad x[8] \quad x[11] \quad \cdots\}$$

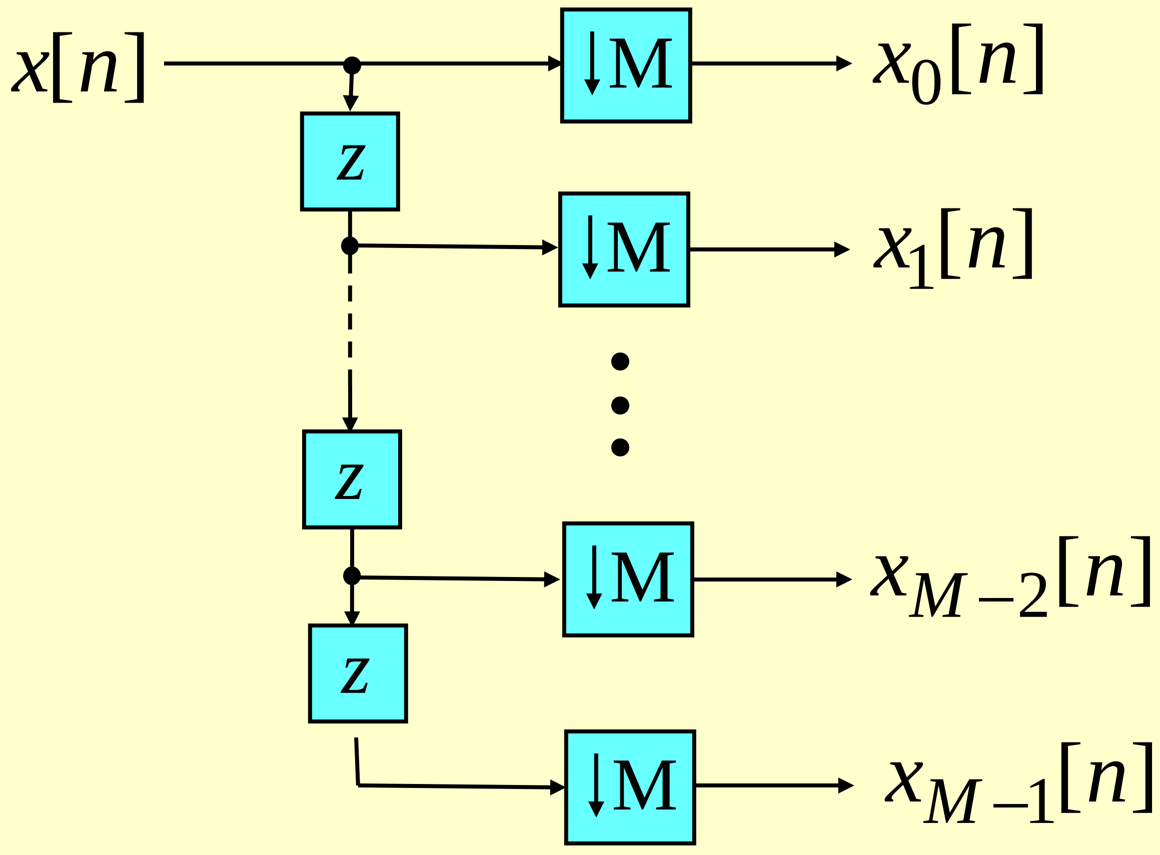
Polyphase Decomposition

- Physical Interpretation – 3-Band Case



Polyphase Decomposition

- Physical Interpretation – General Case



Polyphase Decomposition

- The z-transform $X(z)$ of a finite or infinite length sequence $\{x[n]\}$ can be expressed as a finite sum of the z-transforms $X_i(z)$ of M subsequences $\{x_i[n]\}$, $i = 0, 1, \ldots, M-1$

Polyphase Decomposition

- The M-band polyphase decomposition of $X(z)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{i=0}^{M-1} X_i(z^M) z^{-i}$$

where

$$X_i(z) = \sum_{n=-\infty}^{\infty} x[Mn+i] z^{-n}, \qquad 0 \le i \le M-1$$

- $X_i(z)$ is the i-th polyphase component of $X(z)$

Polyphase Decomposition

- The polyphase decomposition can be written in matrix form as

$$X(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-(M-1)} \end{bmatrix} \begin{bmatrix} X_0(z^M) \\ X_1(z^M) \\ \vdots \\ X_{M-1}(z^M) \end{bmatrix}$$

$$= \mathbf{e}(z) \cdot \mathbf{X}(z^M)^T$$

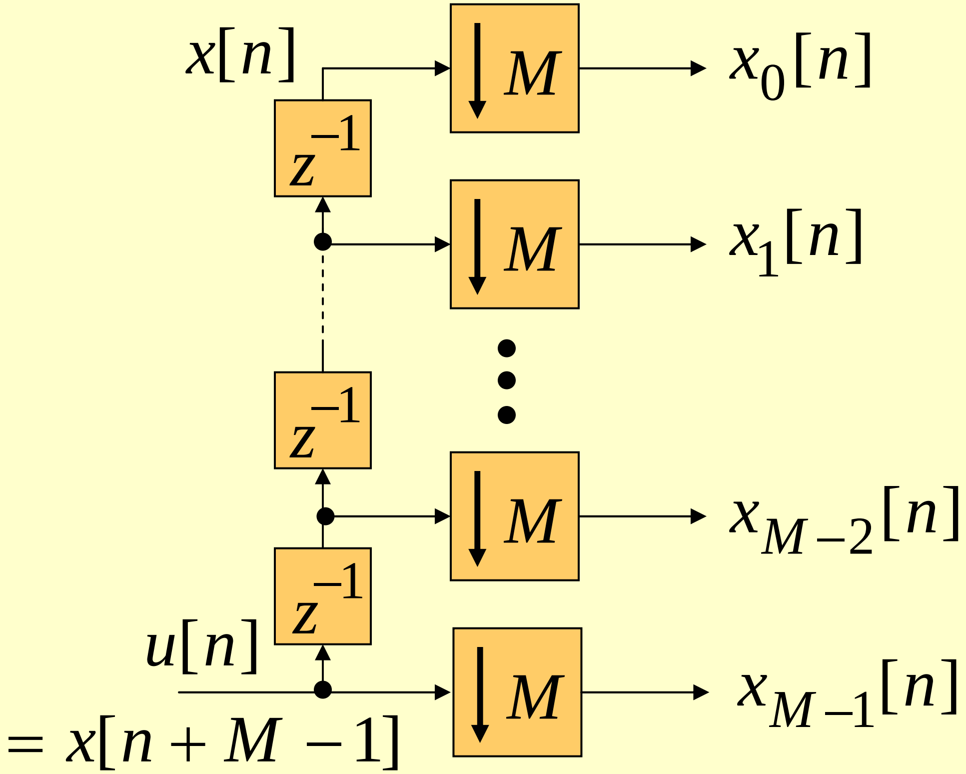
where

$$\mathbf{e}(z) = [1 \quad z^{-1} \quad \cdots \quad z^{-(M-1)}]$$

$$\mathbf{X}(z) = \begin{bmatrix} X_0(z) & X_1(z) & \cdots & X_{M-1}(z) \end{bmatrix}$$

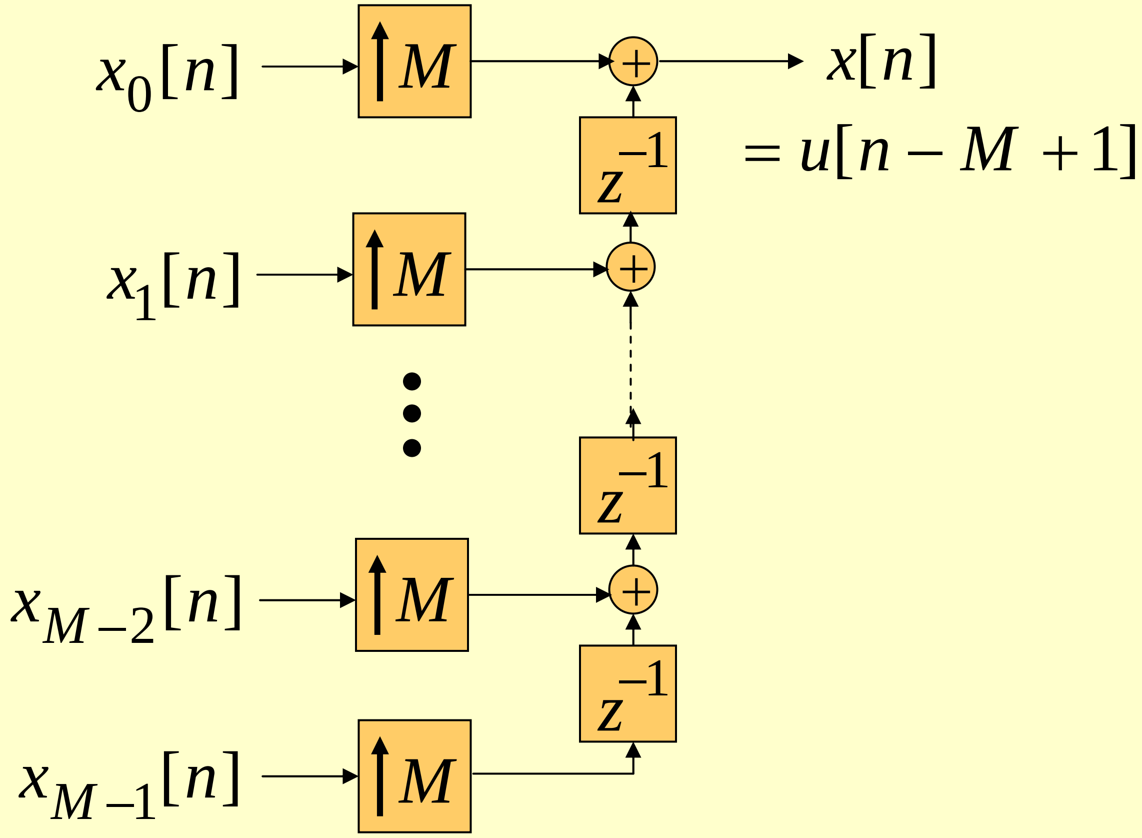
Polyphase Decomposition

- Physical interpretation



Polyphase Decomposition

- Reconstruction of original sequence



Polyphase Decomposition

- The sequence $x[n]$, i.e., a delayed version of the input sequence $u[n]$, can be developed from the M-sub-sequences $x_i[n]$ by up-sampling each subsequence by a factor of M and then interleaving the outputs of the up-samplers

Structural Subband Decomposition

- The structural subband decomposition of $X(z)$ is given by

$$X(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-(M-1)} \end{bmatrix} \mathbf{T} \begin{bmatrix} V_0(z^M) \\ V_1(z^M) \\ \vdots \\ V_{M-1}(z^M) \end{bmatrix}$$

where $\mathbf{T} = [t_{i,j}]$ is an $M \times M$ nonsingular matrix

Structural Subband Decomposition

- The structural subband decomposition is thus a generalization of the polyphase decomposition
- The functions $V_k(z)$ are called the structural subband components or generalized polyphase components of $X(z)$

Structural Subband Decomposition

- Relation between the polyphase components $X_i(z)$ and the structural sub-band components $V_i(z)$ are given by

$$\begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{M-1}(z) \end{bmatrix} = \mathbf{T}^{-1} \begin{bmatrix} X_0(z) \\ X_1(z) \\ \vdots \\ X_{M-1}(z) \end{bmatrix}$$

Structural Subband Decomposition

- If $v_i[n]$ denotes the inverse z-transform of $V_i(z)$, then it follows that

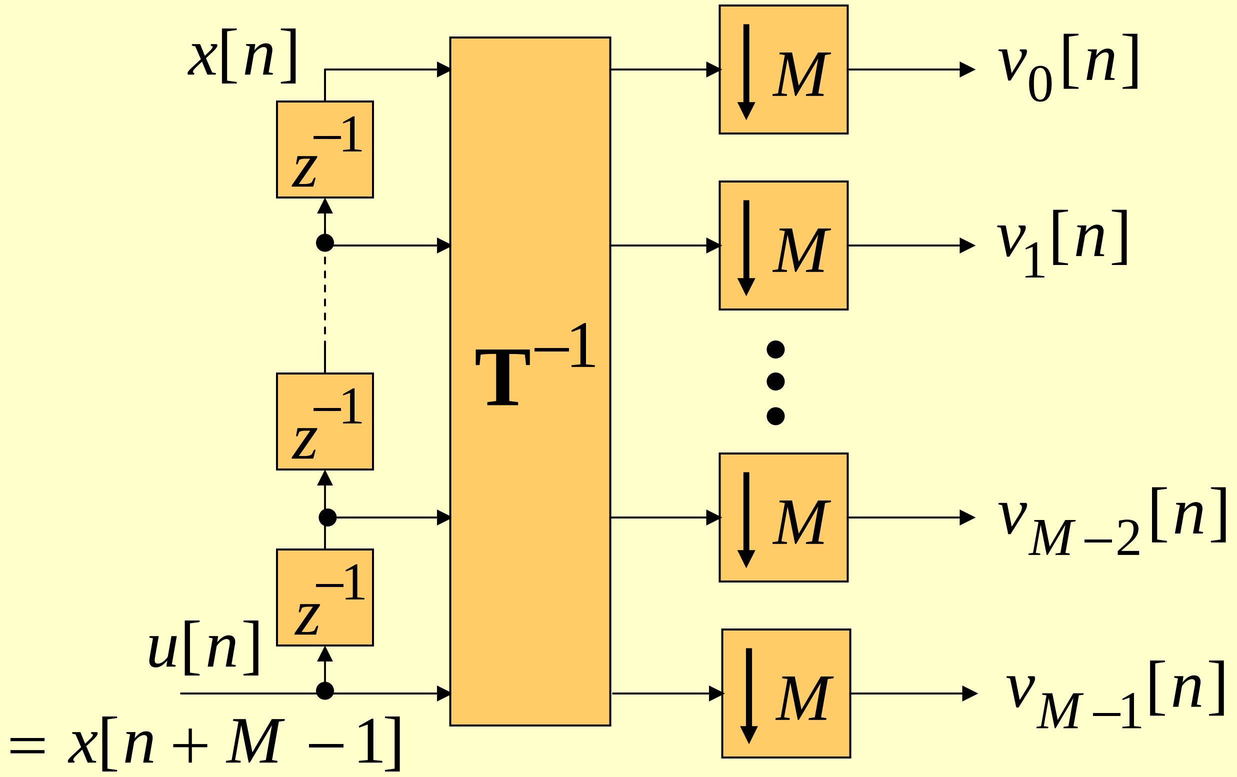
$$v_i[n] = \sum_{\ell=0}^{M-1} \tilde{t}_{i,\ell} x_\ell[n], \quad 0 \le i \le M-1$$

where $\tilde{t}_{i,\ell}$ is the (i,ℓ)-th element of $\mathbf{T}^{-1}$

- The structural sub-band subsequences $v_i[n]$ are basically given by a linear combination of the polyphase sub-sequences $x_i[n]$

Structural Subband Decomposition

- Physical interpretation



Structural Subband Decomposition

- Likewise, the polyphase subsequences $x_i[n]$ can be recovered by a linear combination of the structural subband subsequences $v_i[n]$ according to

$$x_i[n] = \sum_{\ell=0}^{M-1} t_{i,\ell} v_\ell[n], \quad 0 \le i \le M-1$$

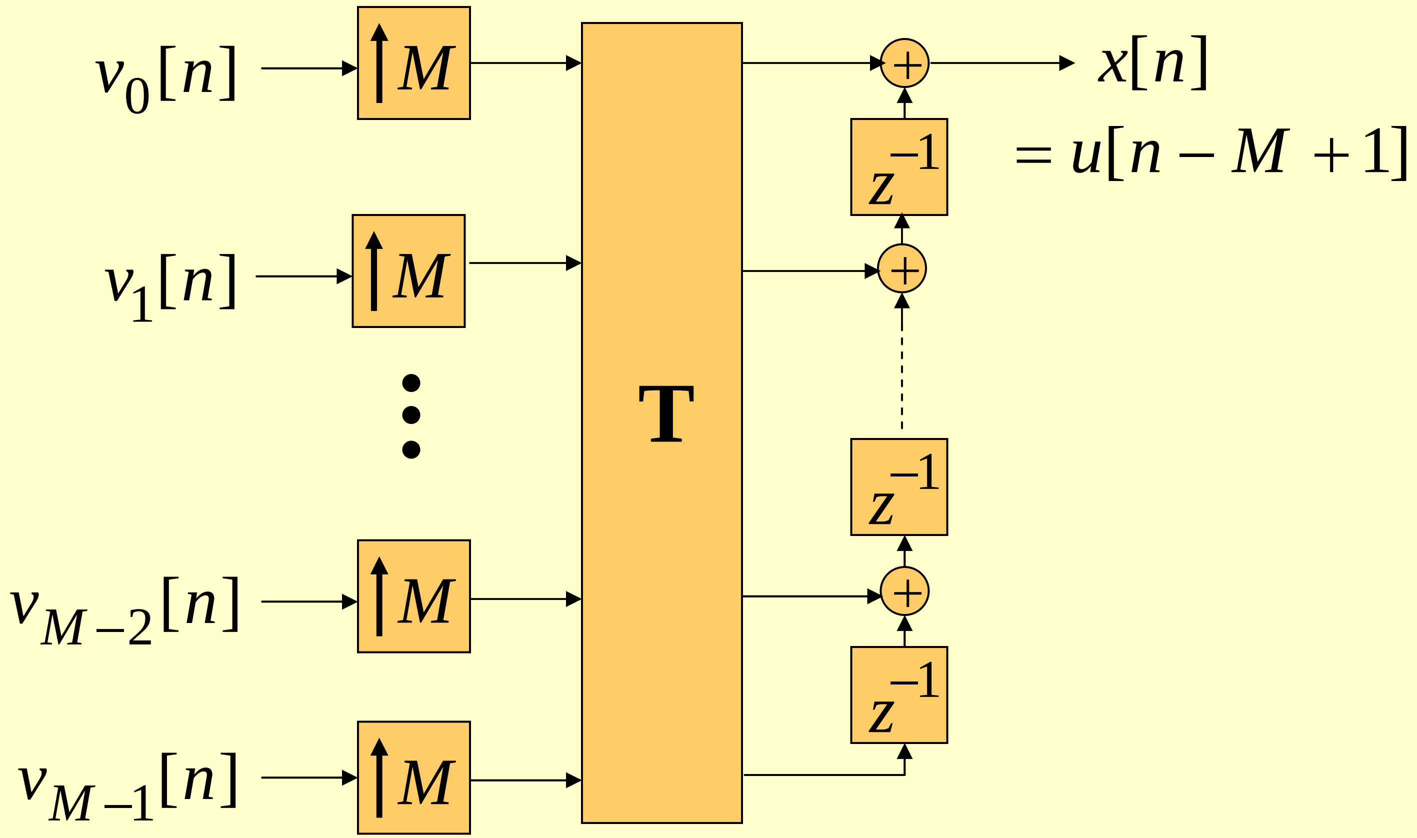
where $t_{i,\ell}$ is the (i,ℓ)-th element of $\mathbf{T}$

Structural Subband Decomposition

- A delayed version of the input $u[n]$ can be developed by first up-sampling the M sub-sequences $v_i[n]$ and then generating the subsequences $\hat{x}_i[n]$ by a linear combination of these up-sampled subsequences, and then interleaving the subsequences $\hat{x}_i[n]$

Structural Subband Decomposition

- Reconstruction of original sequence



Structural Subband Decomposition

- The digital filter structure generating the structural subband sequences can be considered as an M-channel analysis filter bank, characterized by M transfer functions contained in the vector

$$\mathbf{H}(z)^T = [H_0(z) \quad H_1(z) \quad \cdots \quad H_{M-1}(z)]^T$$

$$= z^{-(M-1)} \mathbf{T}^{-1} \mathbf{e}(z^{-1})^T$$

Structural Subband Decomposition

- The digital filter structure forming the reconstructed sequence from the structural subband sequences can be considered as an M-channel synthesis filter bank, characterized by M transfer functions contained in the vector

$$\mathbf{G}(z) = [G_0(z) \quad G_1(z) \quad \cdots \quad G_{M-1}(z)]$$
$$= \mathbf{e}(z) \cdot \mathbf{T}$$

Subband Matrix

- The transfer functions $H_i(z)$ and $G_i(z)$ have bandpass frequency responses for a suitably chosen subband matrix **T**
- Depending on the application, the matrix **T** can have various forms
- To be useful in practice, the matrix **T** should be simple, if possible, both in terms of its elements and its structure

Subband Matrix

- Structural simplicity is inherent in the DFT matrix $\mathbf{W}_M$, which can be efficiently implemented using well known FFT methods
- Here, the channel frequency responses have $\sin\omega/\omega$ form, providing at least some frequency selectivity

Subband Matrix

- However, the elements of $\mathbf{T} = \mathbf{W}_M$ are given by

$$t_{i,\ell} = W_M^{i\ell} = e^{-j2\pi i\ell/M}, \qquad 0 \le i, \ell \le M-1$$

requiring conplex multiplications, choice of $\mathbf{T} = \mathbf{W}_M$ could also be advisable if only very few sub-bands are desired

Subband Matrix

- For example, for $M = 4$, we have

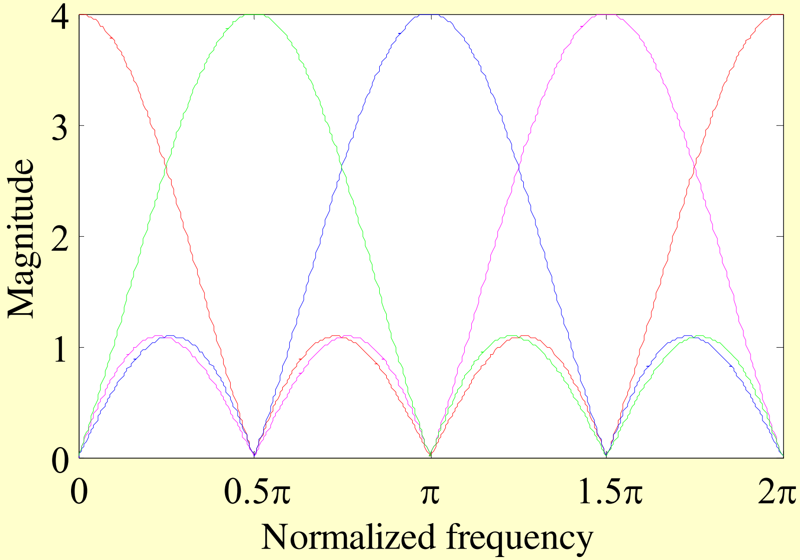
$$\mathbf{T} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\mathbf{T}^{-1} = \frac{1}{4}\mathbf{T}^{*} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

which do not require any true multiplications

Subband Matrix

- The corresponding magnitude responses are shown below



Subband Matrix

- Both structural and element-wise simplicities are inherent in the $M \times M$ Hadamard matrix $\mathbf{R}_M$, given by

$$\mathbf{R}_M = \underbrace{\mathbf{R}_2 \otimes \mathbf{R}_2 \otimes \cdots \otimes \mathbf{R}_2}_{\frac{M}{2} \text{ terms}}$$

where $\mathbf{R}_2$ is the 2×2 Hadamard matrix

$$\mathbf{R}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and $\otimes$ is the Kronecker roduct

Subband Matrix

- From the definition it follows that the order M of the Hadamard matrix must be a power-of-2, i.e. $M = 2^{\mu}$
- It can be shown that

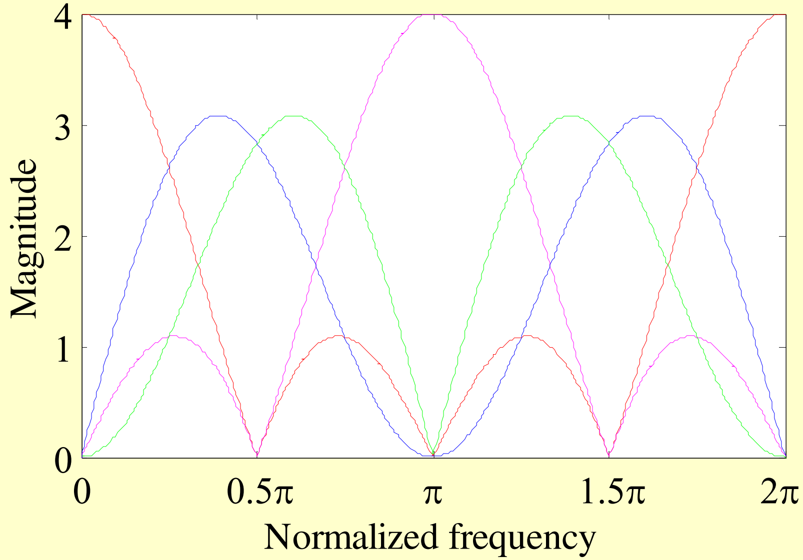
$$\mathbf{R}_M^{-1} = \frac{1}{M}\mathbf{R}_M$$

- For $M = 4$,

$$\mathbf{T} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Subband Matrix

- The corresponding magnitude responses are shown below



- Somewhat higher frequency selectivity of the bandpass responses have been obtained with a slight modified form of the matrix

Subband Discrete Transforms

- An interesting application of the structural subband decomposition concept is in the approximate, but fast, computation of dominant discrete-transform samples
- Two particular discrete transforms considered here are:

  (1) Subband discrete Fourier transform,

  (2) Subband discrete cosine transform
- The concept can be extended to other types of transforms and higher dimensions

Subband Discrete Fourier Transform

- The N-point DFT $X[k]$ of a length-N sequence $x[n]$ is given by the N samples of its z-transform $X(z)$ evaluated on the unit circle at N equally spaced points,

$$X[k] = X(z)\big|_{z=W_N^{-k}} = \sum_{n=0}^{N-1} x[n] W_N^{nk},$$

$$0 \le k \le N-1$$

where $W_N = e^{-j2\pi/N}$

Subband Discrete Fourier Transform

- From the M-band polyphase decomposition of $X(z)$

$$X(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-(M-1)} \end{bmatrix} \begin{bmatrix} X_0(z^M) \\ X_1(z^M) \\ \vdots \\ X_{M-1}(z^M) \end{bmatrix}$$

with $P = N/M$ integer, it follows that

Subband Discrete Fourier Transform

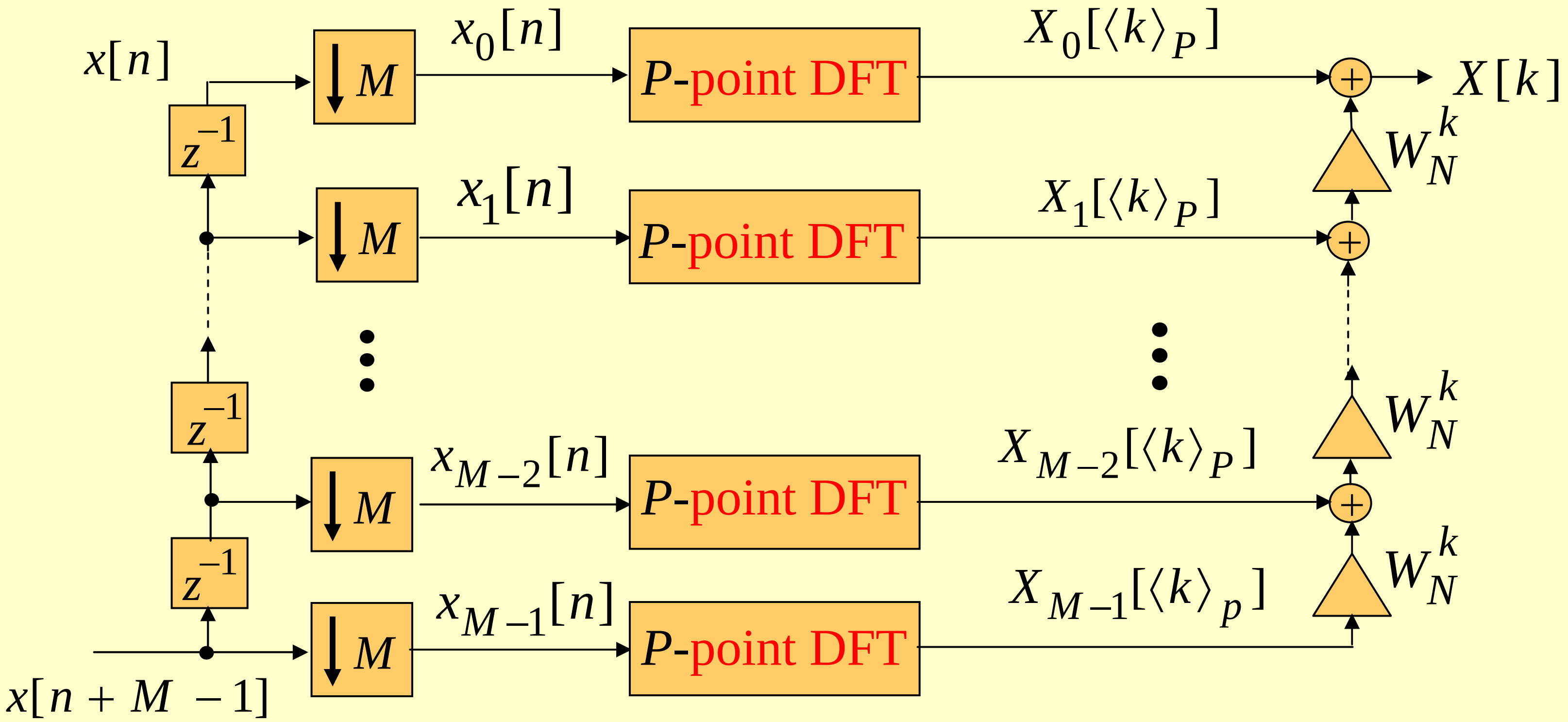
- the DFT samples can alternately be expressed in the form

$$X[k] = \begin{bmatrix} 1 & W_N^k & \cdots & W_N^{(M-)k} \end{bmatrix} \begin{bmatrix} X_0[\langle k \rangle_P] \\ X_1[\langle k \rangle_P] \\ \vdots \\ X_{M-1}[\langle k \rangle_P] \end{bmatrix}$$

where $\langle k \rangle_P = k \,\mathrm{modulo}\, P$ and $X_i[k]$ is the P-point DFT of the polyphase component $x_i[n]$

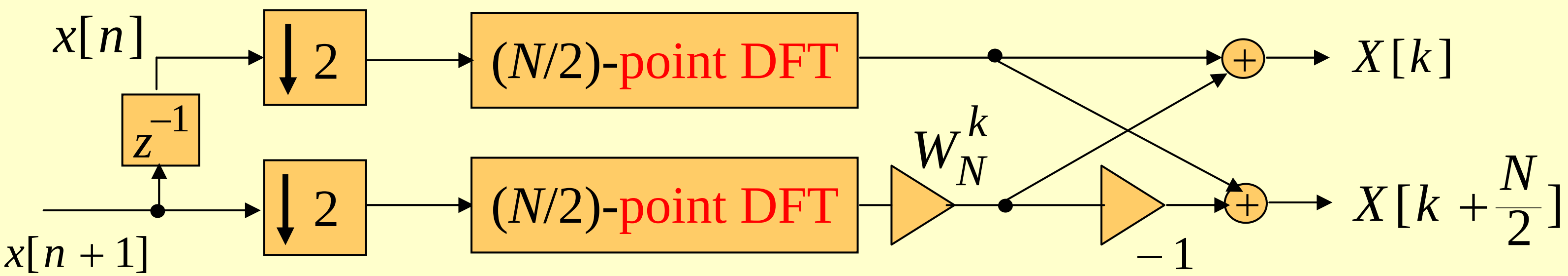
Subband Discrete Fourier Transform

- Physical interpretation



Subband Discrete Fourier Transform

- For $M = 2$, we have



which describes the final twiddle-factor/ butterfly structure of a radix-2, decimation-in-time Cooley-Tukey (CT)-FFT

Subband Discrete Fourier Transform

- From the M-band structural sub-band decomposition of $X(z)$

$$X(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-(M-1)} \end{bmatrix} \mathbf{T} \begin{bmatrix} V_0(z^M) \\ V_1(z^M) \\ \vdots \\ V_{M-1}(z^M) \end{bmatrix}$$

with $P = N/M$ integer, it follows that

Subband DFT

- the DFT samples can alternately be expressed in the form

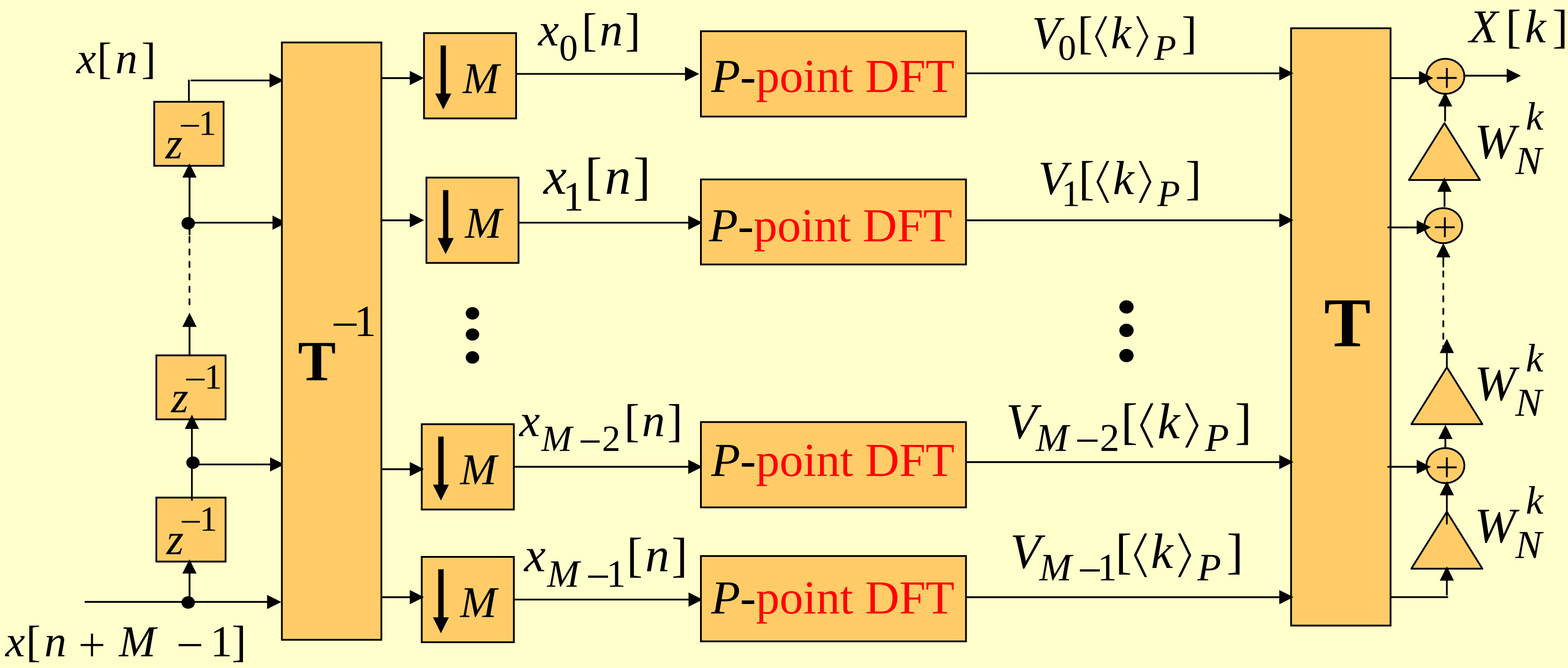
$$X[k] = \begin{bmatrix} 1 & W_N^k & \cdots & W_N^{(M-)k} \end{bmatrix} \cdot \mathbf{T} \cdot \begin{bmatrix} V_0[\langle k \rangle_P] \\ V_1[\langle k \rangle_P] \\ \vdots \\ V_{M-1}[\langle k \rangle_P] \end{bmatrix}$$

where $V_i[k]$ is the P-point DFT of the i-th structural subband component $v_i[n]$

- This is the general form of the subband discrete Fourier transform (SB-DFT)

Subband DFT

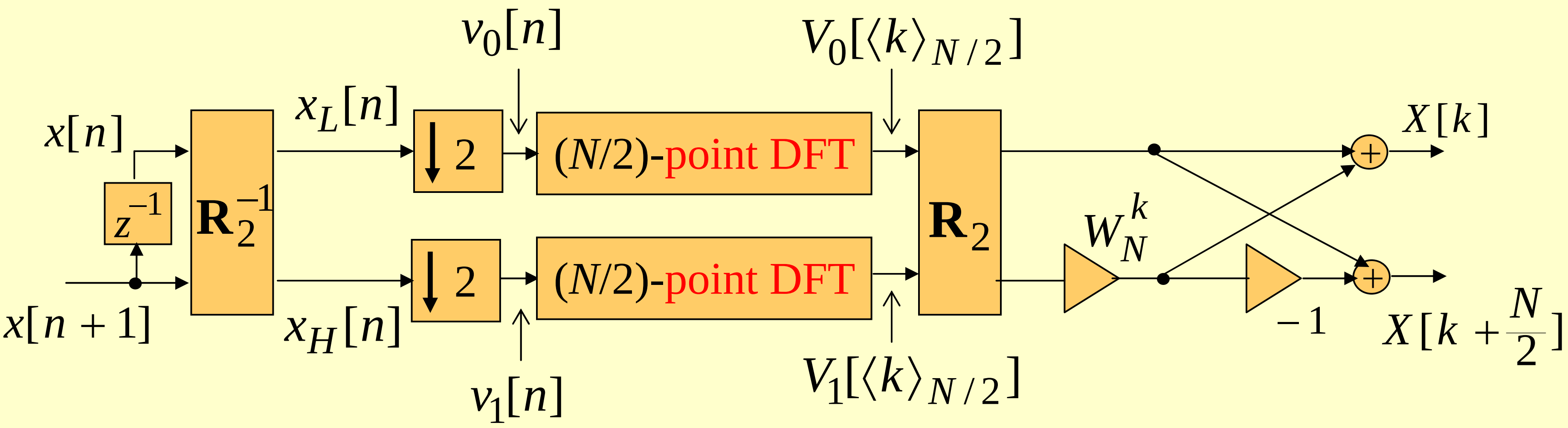
- Physical interpretation



Subband DFT

- For $M = 2$ with $\mathbf{T} = \mathbf{R}_2$, we have

$$X[k] = (1 + W_N^k) \cdot V_0[k] + (1 - W_N^k) \cdot V_1[k], \quad 0 \le k \le \frac{N}{2}$$



- Note: $x_L[n]$ is a lowpass signal, whereas,
  $x_H[n]$ is a highpass signal

Subband DFT

- For $N = 2^{\nu}$, if the decimation by $M = 2$ is repeated $\nu - 1$ times, a full-band SB-DFT algorithm results

- For $\mathbf{T} = \mathbf{R}_M$, it contains a length-$N$ fast Hadamard transform

Subband DFT

- The number of multiplications required is equal to $\frac{N}{2} \cdot \log_2 N$, same as in the CT-FFT algorithm
- However, there are $2N(\log_2 N - 1)$ more additions than that required in the CT-FFT due to the implementation of $\mathbf{R}_M^{-1}$
- In the general case with a different sub-band matrix $\mathbf{T}$, additional multiplications may arise

Subband DFT

- If the signal is a priori band-limited to a cut-off frequency $\omega_c \leq \pi / M$, it may be simply down-sampled by a factor of M, and only N/M values feed a shorter FFT: the polyphase approach is then applicable

- If, however, the signal is not strictly band-limited, aliasing occurs

Subband DFT

- In the subband approach aliasing effects are reduced by the pre-filters $H_i(z)$
- Then, if the reduced aliasing is acceptable, branches can be dropped by pruning the SB-DFT and obtain approximate values of the dominant DFT samples

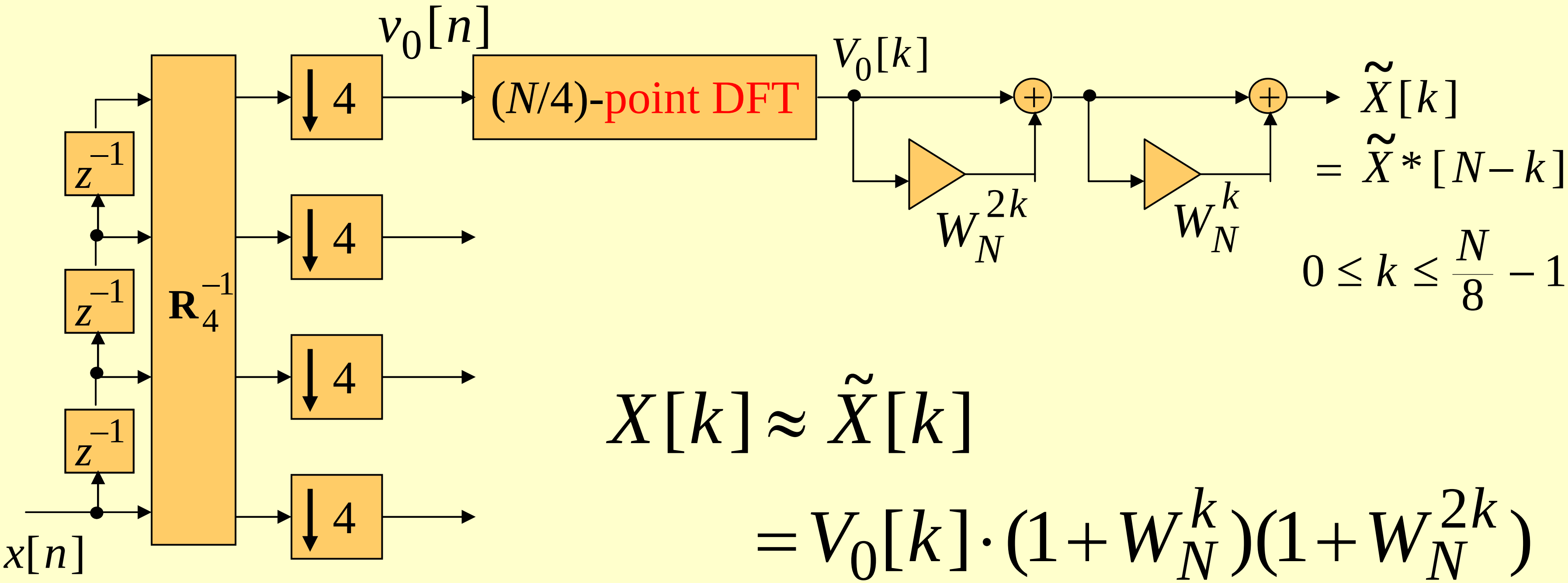
Subband DFT

- For example, if 1 band in an M-band subband decomposition is dominant, $(M-1)$ branches can be dropped and calculate a standard CT-FFT of length N/M of one decimated signal

- For an 8-band analysis, only 40% of the CT-FFT computer time is needed

Subband DFT

- Approximate SB-DFT calculation with $M = 4$, $\mathbf{T} = \mathbf{R}_4$, and dropping of 3 out of 4 bands



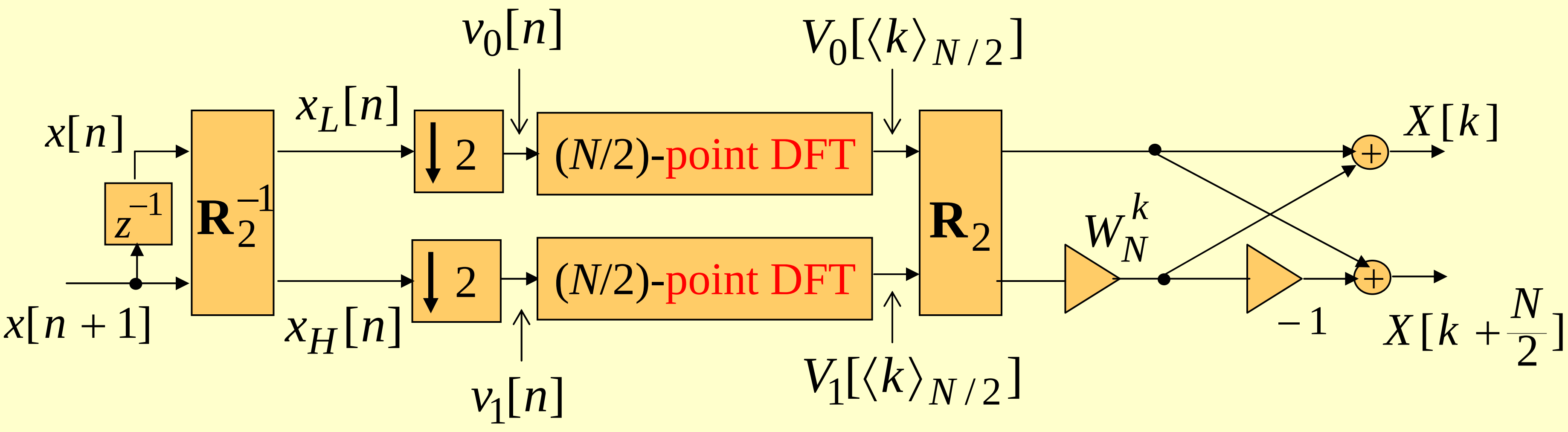
$$X[k] \approx \tilde{X}[k]$$

$$= V_0[k] \cdot (1 + W_N^k)(1 + W_N^{2k})$$

$$0 \le k \le \frac{N}{8} - 1$$

Subband DFT

- Adaptive band selection in the case of Hadamard transform based sub-band DFT
- Based on averaged (signs of) differences between $|v_0[n]|$ and $|v_1[n]|$ in the 2-band DFT computation scheme shown below



Subband DFT

- In the general case of $M > 2$, the method is based on averaged (signs of) differences between corresponding subband component pairs
- The online estimation causes only a minor loss of computational advantage gained by the subband calculation

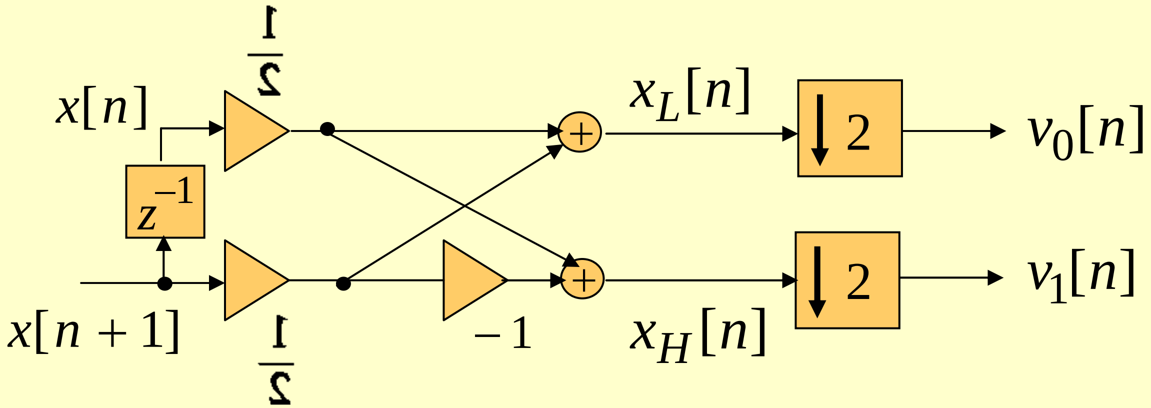
Subband Discrete Cosine Transform

- The structural subband decomposition concept has also been applied to the approximate, but efficient, computation of the dominant samples of the DCT
- One of the most common forms of the DCT of a length-N sequence $x[n]$, with N even, is given by

$$C[k] = \sum 2x[n]\cos\left(\frac{(2n+1)\pi k}{2N}\right), \qquad 0 \le k \le N-1$$

Subband DCT

- By applying the subband processing to $x[n]$



we can write

$$C[k] = 2\cos\left(\frac{\pi k}{2N}\right)\bar{C}_0[k] + 2\sin\left(\frac{\pi k}{2N}\right)\bar{S}_0[k],$$

$$0 \leq k \leq N-1$$

Subband DCT

where

$$\bar{C}_0[k] = \begin{cases} C_0[k], & 0 \le k \le \frac{N}{2} - 1 \\ 0, & k = \frac{N}{2} \\ -C_0[N-k], & \frac{N}{2} + 1 \le k \le N - 1 \end{cases}$$

with $C_0[k]$ denoting the (N/2)-point DCT (discrete cosine transform) of $v_0[n]$

Subband DCT

and

$$\bar{S}_1[k] = \begin{cases} S_1[k], & 0 \le k \le \frac{N}{2} - 1 \\ 2 \sum_{n=0}^{(N-2)/2} (-1)^n x_1[n], & k = \frac{N}{2} \\ S_1[N-k], & \frac{N}{2} + 1 \le k \le N - 1 \end{cases}$$

with $S_1[k]$ denoting the (N/2)-point DST (discrete sine transform) of $v_1[n]$

Subband DCT

- The computation of the N-point DCT $C[k]$ requiring the computation of an $(N/2)$-point DCT $C_0[k]$ and an $(N/2)$-point DST $S_1[k]$ has been referred to as the subband DCT
- The above process can be continued to decompose the sub-sequences $v_0[n]$ and $v_1[n]$, provided $N/2$ is an even integer
- The process terminates when the final subsequences are of length 2

Subband DCT

- By exploiting the spectral contents of the subsequences, an efficient DCT algorithm can be developed
- For example, if $x[n]$ is known to have most of its energy in the low frequencies, a reasonable approximation to $C[k]$ can be obtained by discarding terms associated with high frequencies

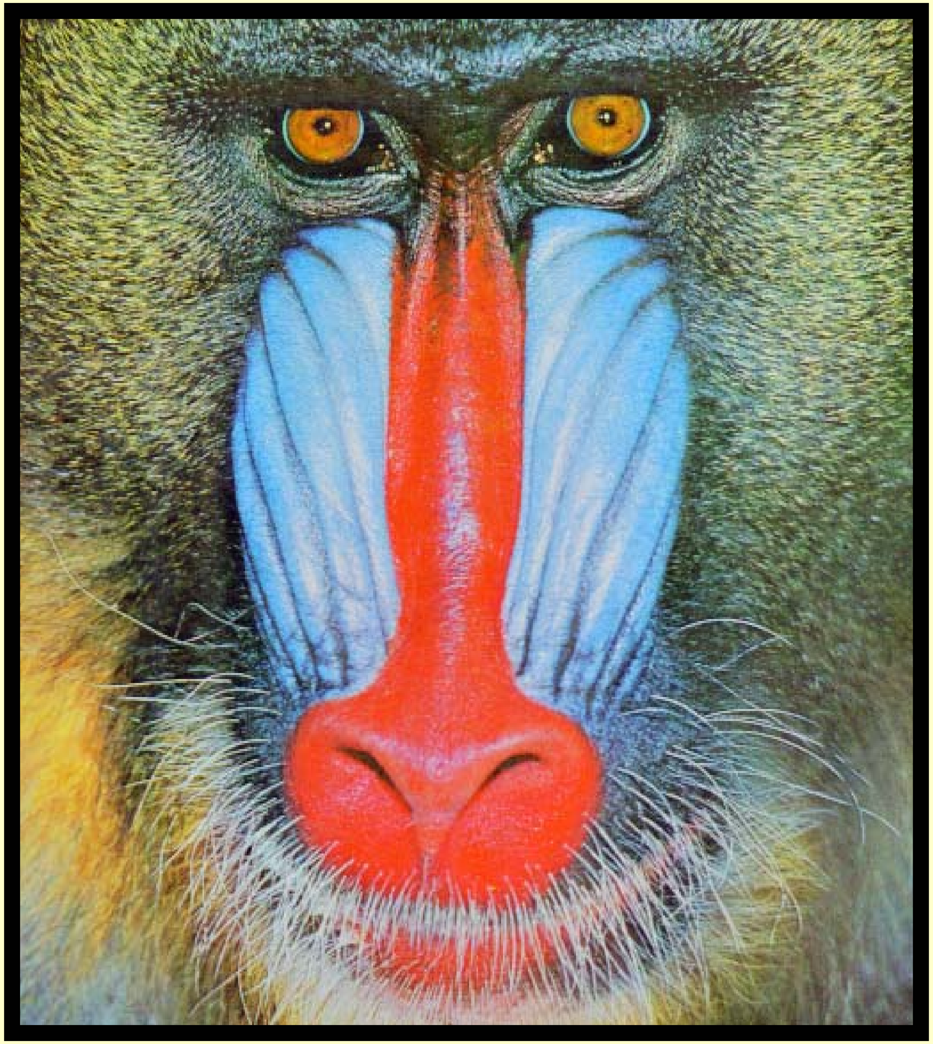
Subband DCT

- The resulting approximation is given by

$$C[k] \cong \begin{cases} 2\cos\left(\dfrac{\pi k}{2N}\right) C_0[k] & 0 \le k \le \dfrac{N}{2} - 1 \\ 0, & \text{otherwise} \end{cases}$$

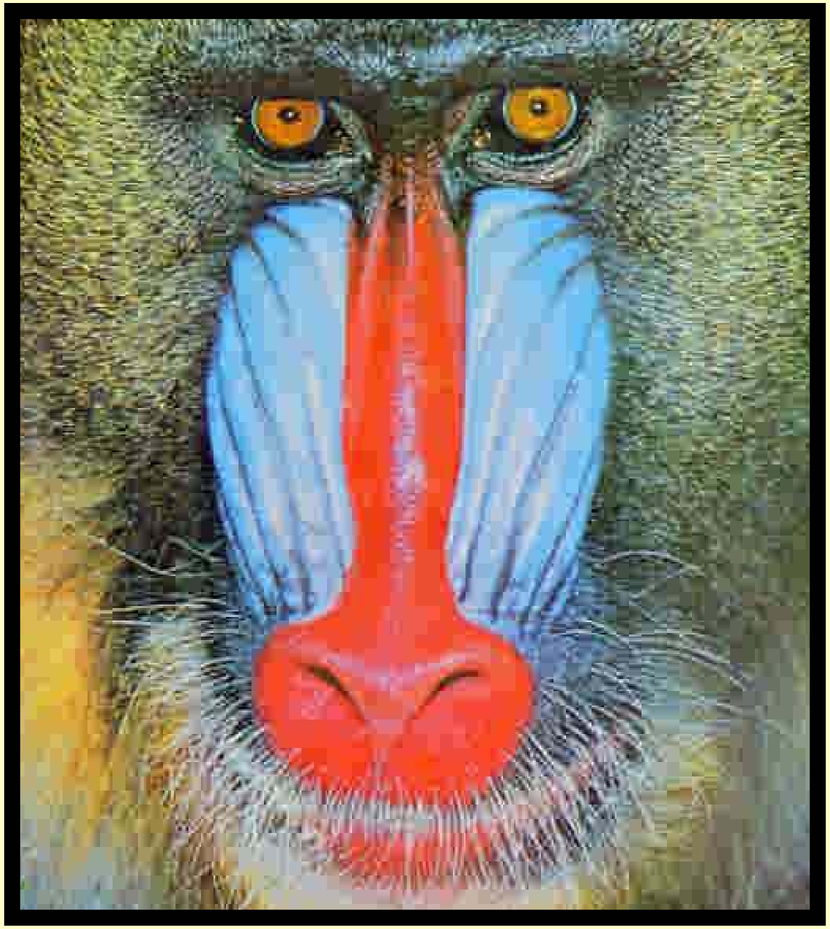
- The SB-DCT concept can be extended to higher dimensions

Image Compression Application

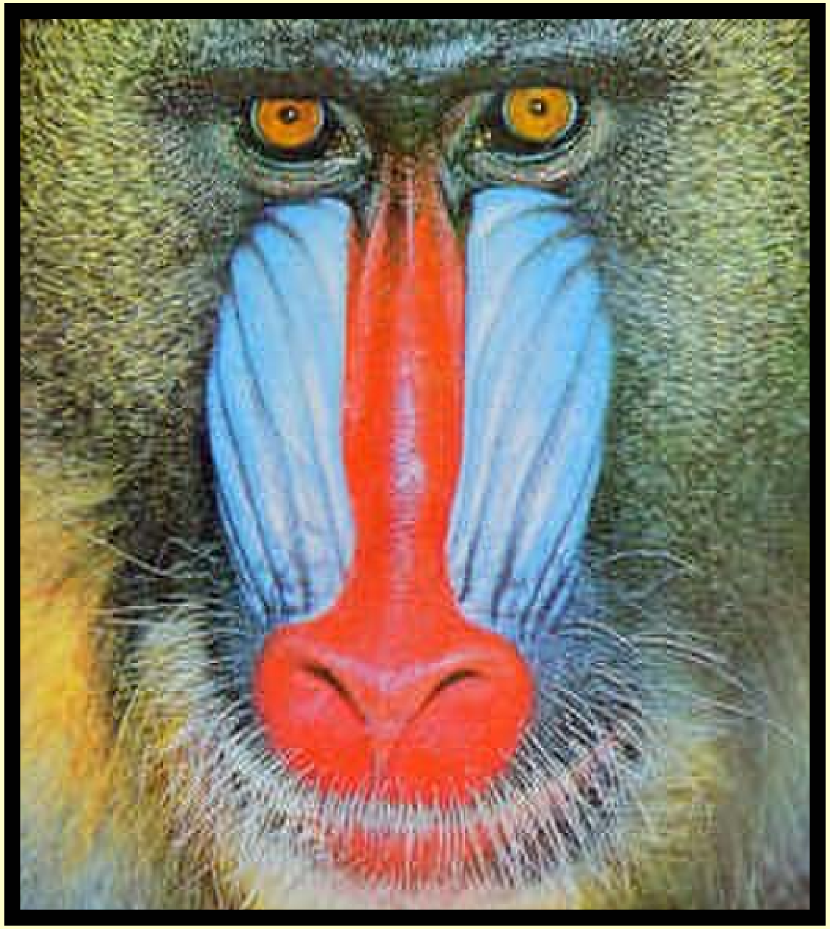


Original BABOON image

Image Compression Application



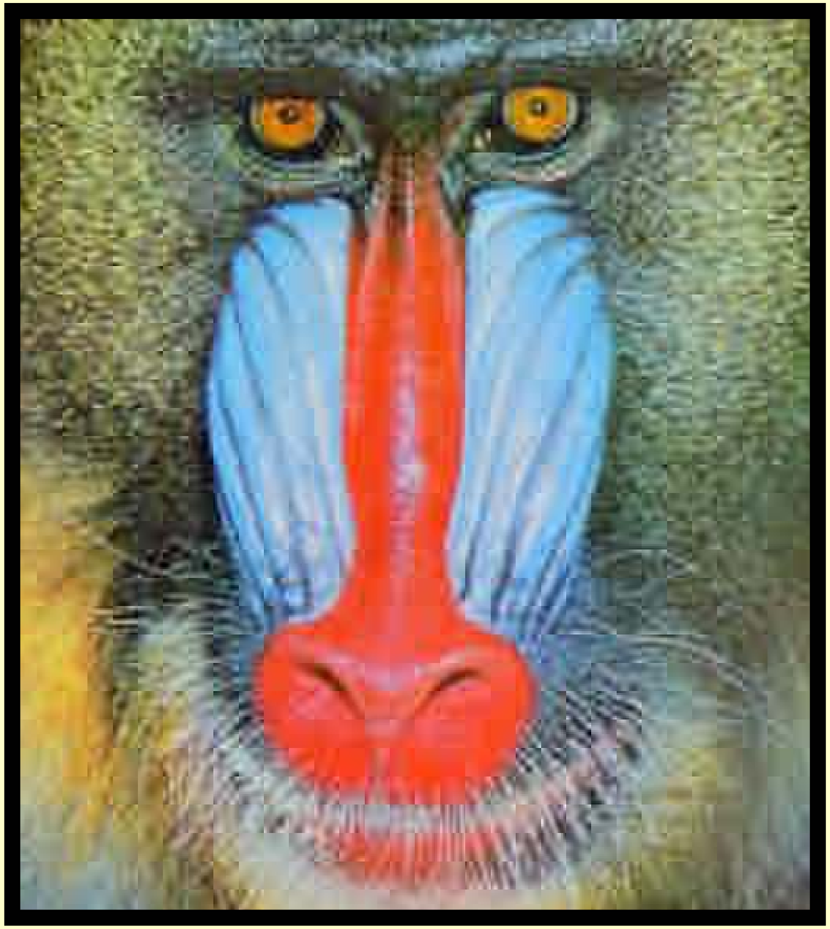
Standard DCT, compr 50



Sub-band DCT, compr 50

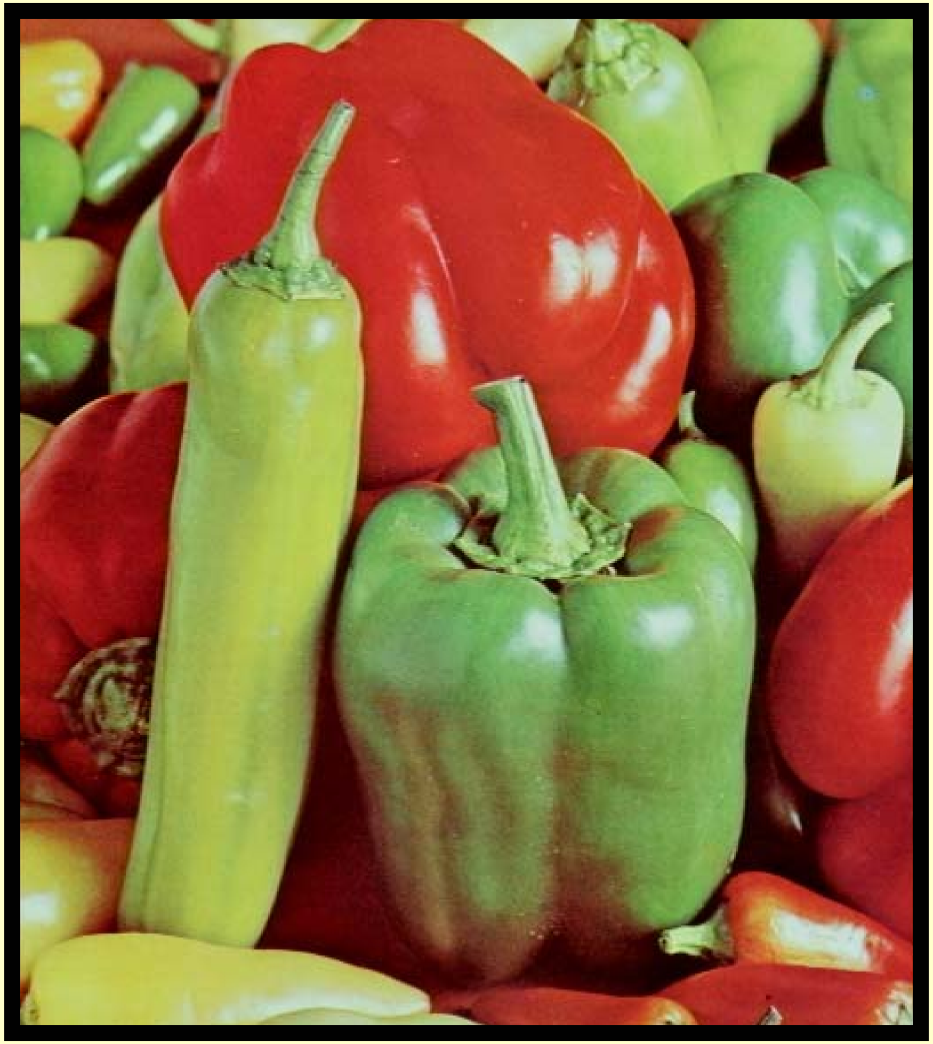
Image Compression Application

Standard DCT, compr 100



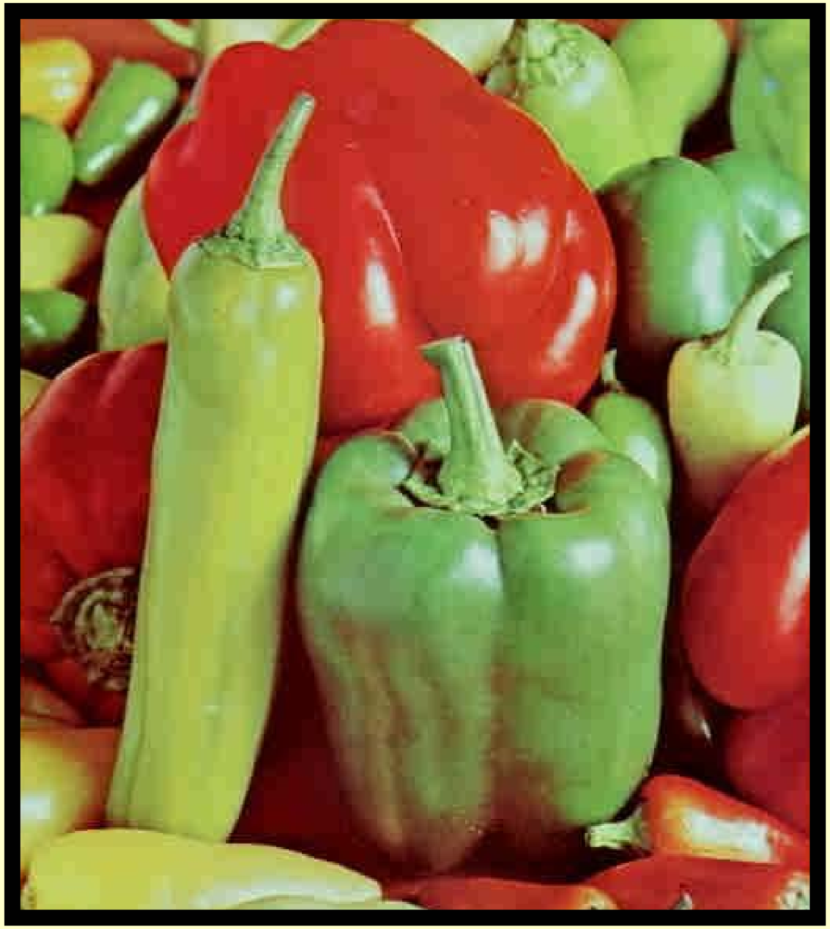
Sub-band DCT, compr 100

Image Compression Application

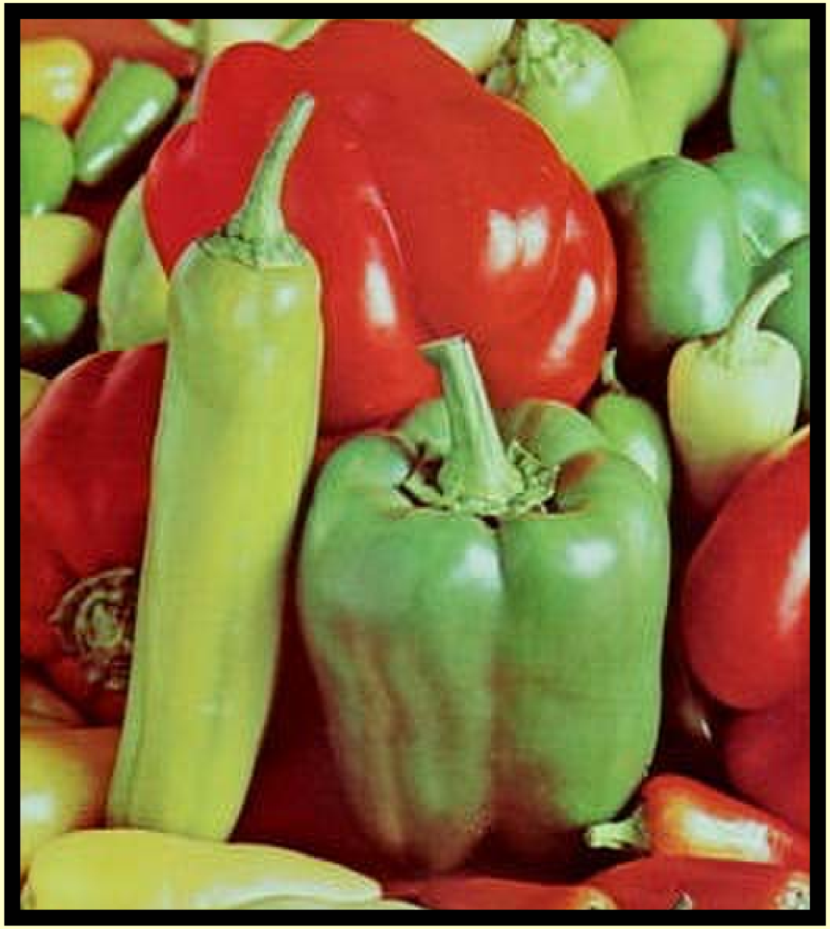


Original PEPPERS image

Image Compression Application

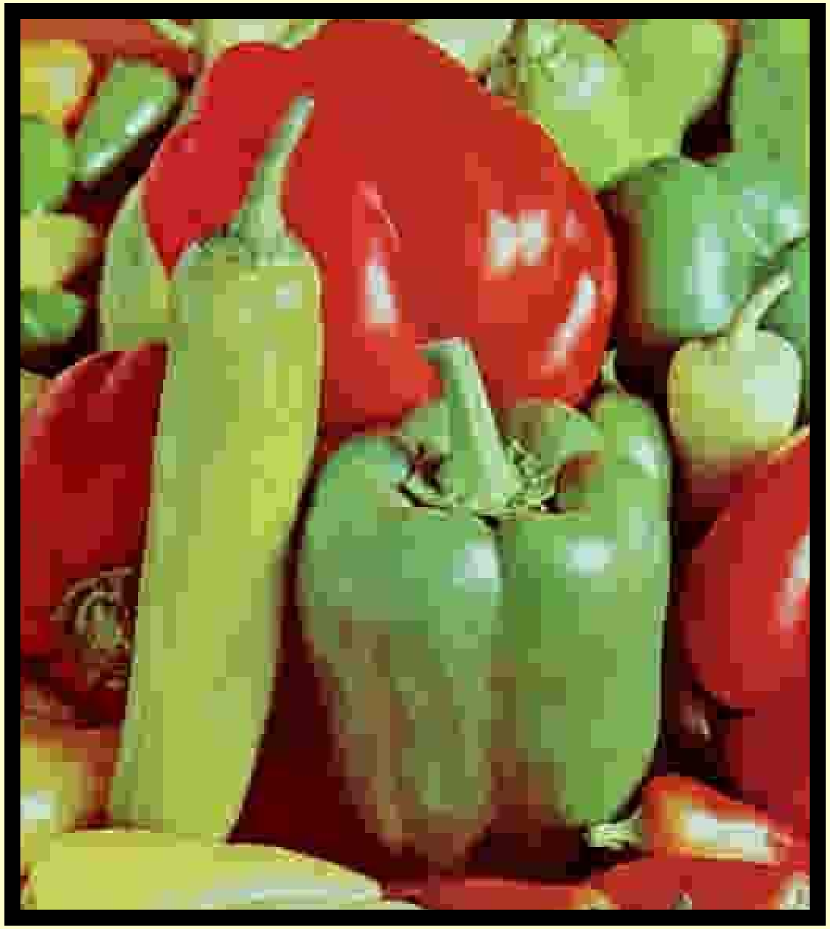


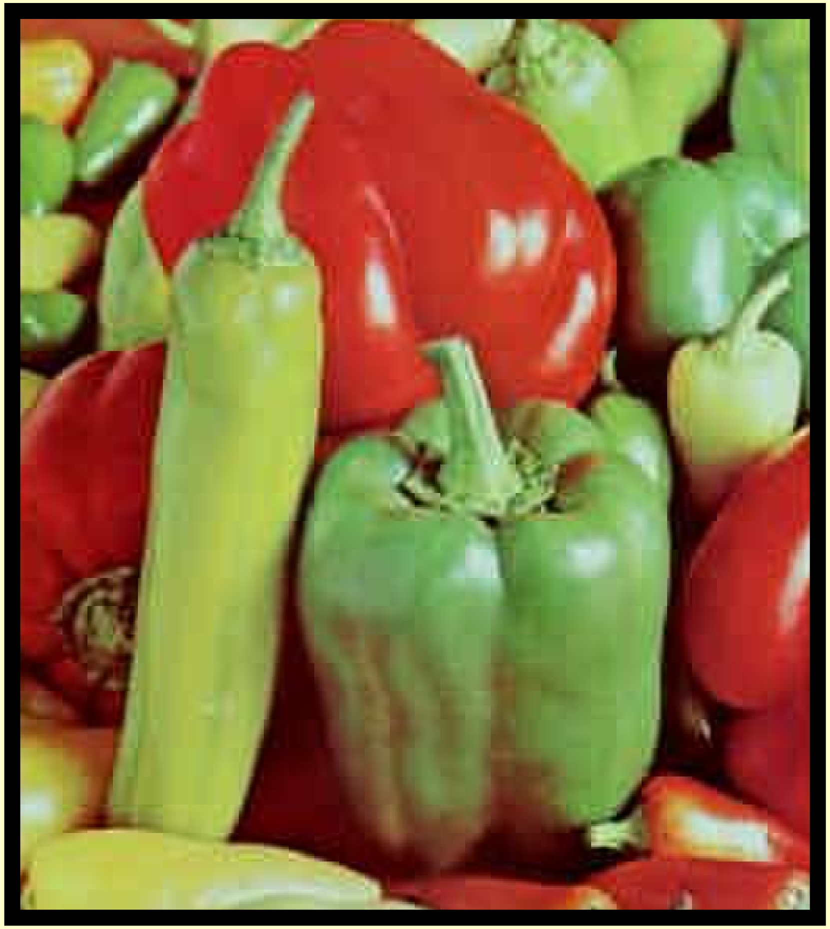
Standard DCT, compr 50



Sub-band DCT, compr 50

Image Compression Application





Standrard DCT, compr 100 Sub-band DCT, compr 100

Efficient FIR Filter Design and Implementation

- Consider an FIR filter $H(z)$ with an impulse response $\{h[n]\}$ of length $N = P \times M$
- By applying the structural subband decomposition to $H(z)$ we arrive at

$$H(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-(M-1)} \end{bmatrix} \mathbf{T} \begin{bmatrix} F_0(z^M) \\ F_1(z^M) \\ \vdots \\ F_{M-1}(z^M) \end{bmatrix}$$

Efficient FIR Filter Design and Implementation

- The M-band structural subband decomposition of $H(z)$ can be alternately expressed as

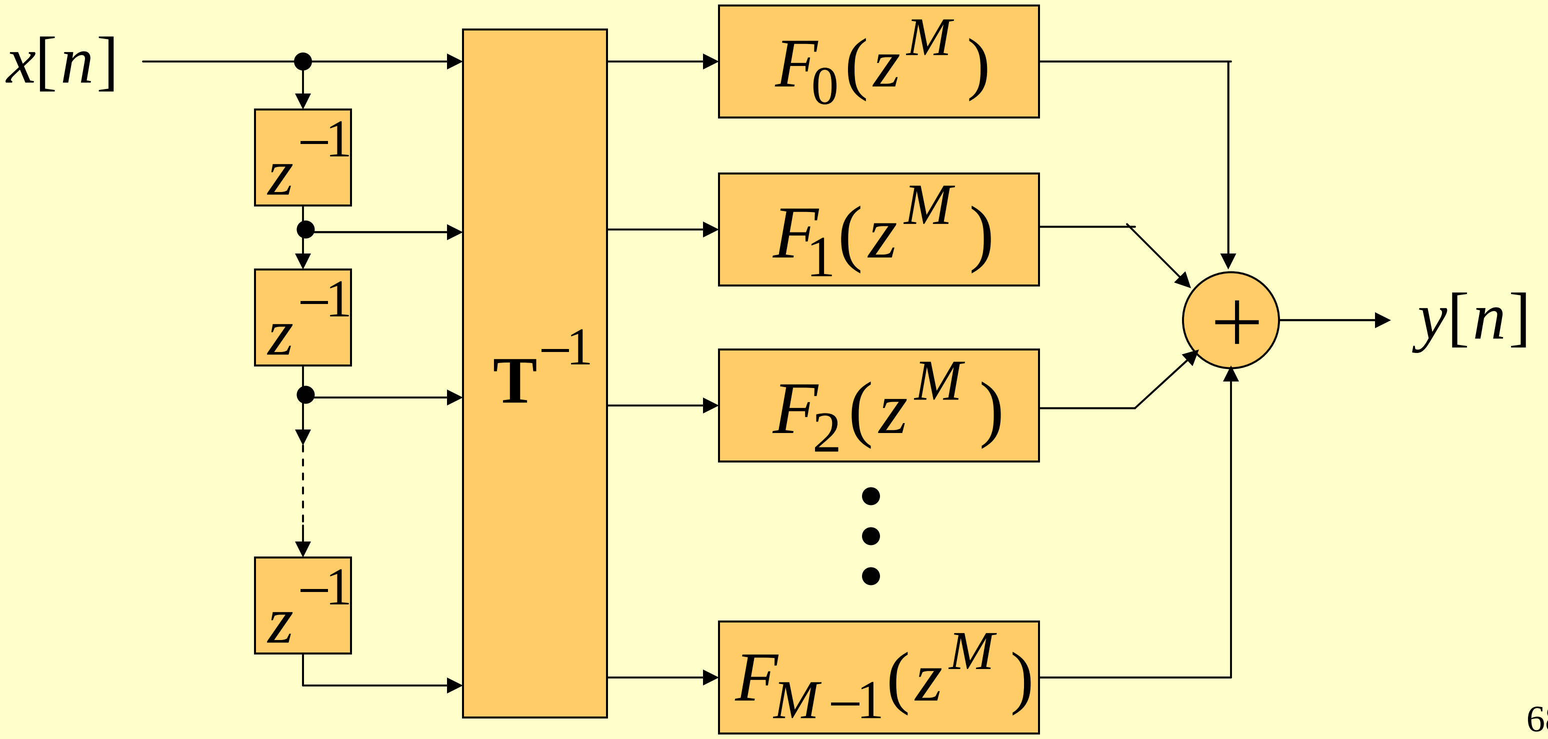
$$H(z) = \sum_{k=0}^{M-1} G_k(z) F_k(z^M)$$

where $G_k(z)$ is given by

$$G_k(z) = \sum_{\ell=0}^{M-1} t_{\ell,k} z^{-j}, \quad 0 \le k \le M-1$$

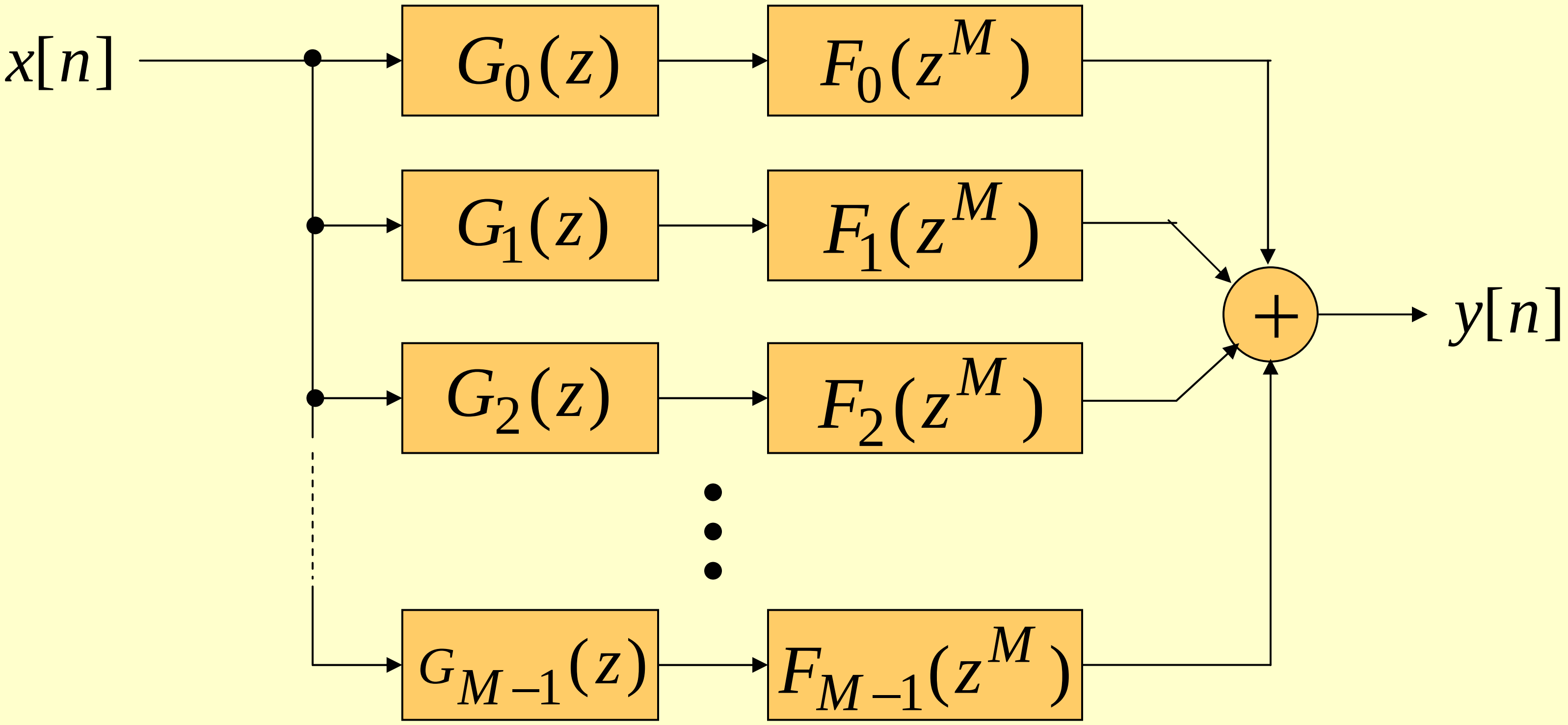
Efficient FIR Filter Implementation

- Realizations of $H(z)$ based on the structural subband decomposition are as follows:



Efficient FIR Filter Implementation

- Parallel IFIR realization



Efficient FIR Filter Implementation

- Thus the second realization can be considered as a generalization of the interpolated FIR (IFIR) structure, where $G_k(z)$ is the interpolator and $F_k(z)$, the shaping filter, is of length $P = N/M$
- Note: Delays in the implementation of the sub-filters $F_k(z^M)$ in both realizations can be shared leading to a canonic realization of the overall structure

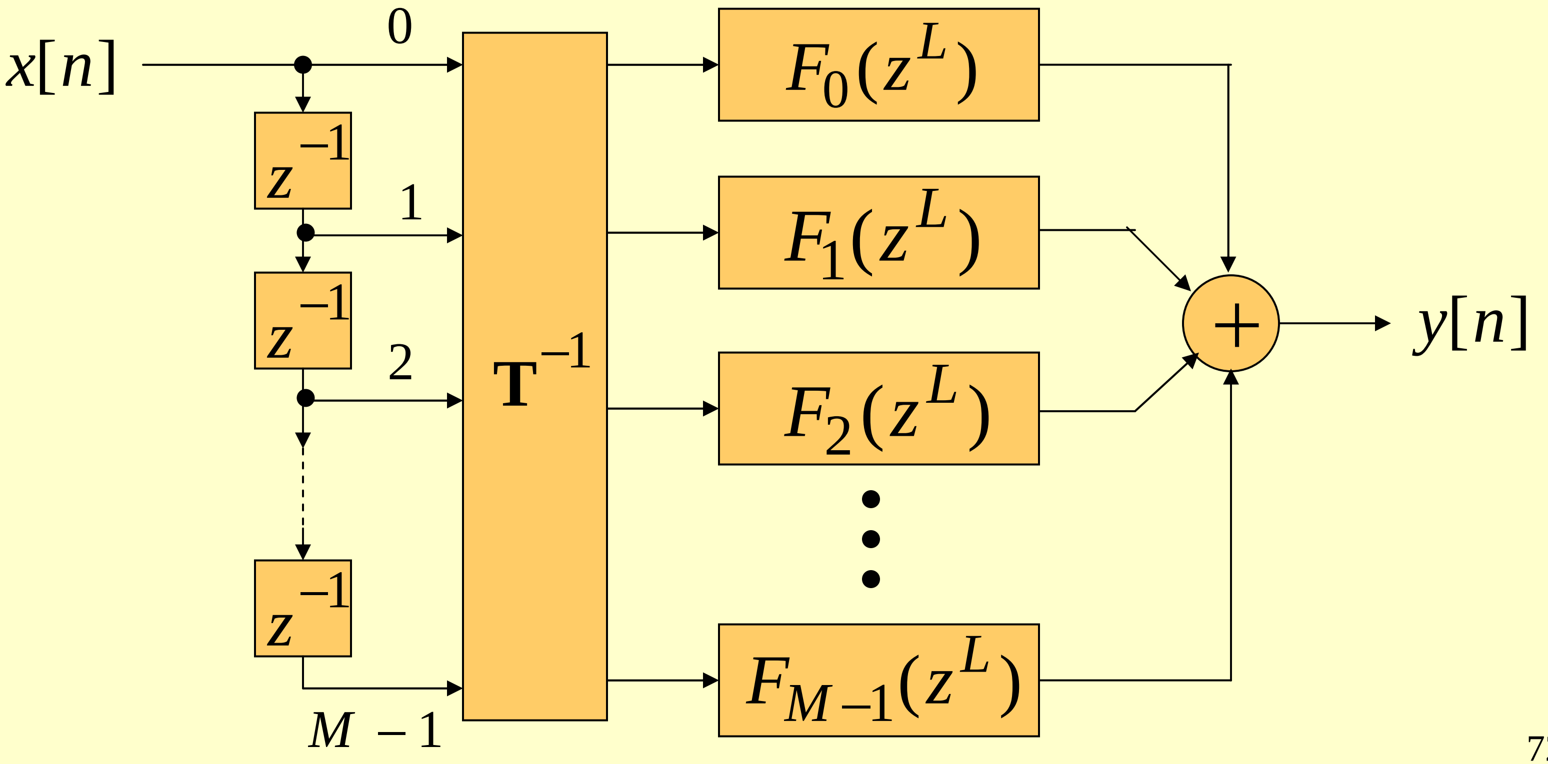
Efficient FIR Filter Implementation

- Further generalization obtained by choosing the number of bands M (i.e. the sub-band transform size) different from the sparsity factor L of the subfilters $F_k(z^L)$

$$H(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-(M-1)} \end{bmatrix} \mathbf{T} \begin{bmatrix} F_0(z^L) \\ F_1(z^L) \\ \vdots \\ F_{M-1}(z^L) \end{bmatrix}$$

Efficient FIR Filter Implementation

- Corresponding realization



Efficient FIR Filter Implementation

- For $L \leq M$, the modified structure can realize any FIR transfer function $H(z)$ of length up to $N = (P-1)L + M$, where P is the length of $F_k(z)$
- Coefficients of $F_k(z)$ are no longer unique, resulting in an infinite number of realizations for a given $H(z)$ with fixed L and M
- For $L < M$, there is an increase in the number of multipliers

Efficient FIR Filter Implementation

- Computational complexity of the overall structure can be reduced by choosing “simple” invertible transform matrices **T** such as the Hadamard matrix

- Each interpolator section is a cascade of μ basic interpolators of the form

Efficient FIR Filter Implementation

- For an M-branch decomposition, the interpolator $G_0(z)$ has a lowpass magnitude response given by

$$\left|G_0(e^{j\omega})\right| = \frac{\sin(M\omega/2)}{\sin(\omega/2)}$$

- The interpolator $G_1(z)$ has a highpass magnitude response given by

$$\left|G_1(e^{j\omega})\right| = \frac{\sin[M(\pi-\omega)/2]}{\sin[(\pi-\omega)/2]}$$

- The remaining interpolators $G_k(z)$ with $k \neq 0,1$ have each a bandpass magnitude response

Efficient FIR Filter Implementation

- Each of the branches thus contributes to the overall response essentially within a "subband" associated with the corresponding interpolator
- For a narrow-band FIR filter, it may be possible to drop branches from the overall structure if these branches do not contribute significantly to the filter's frequency response, thus leading to a computationally efficient realization

Efficient FIR Filter Implementation

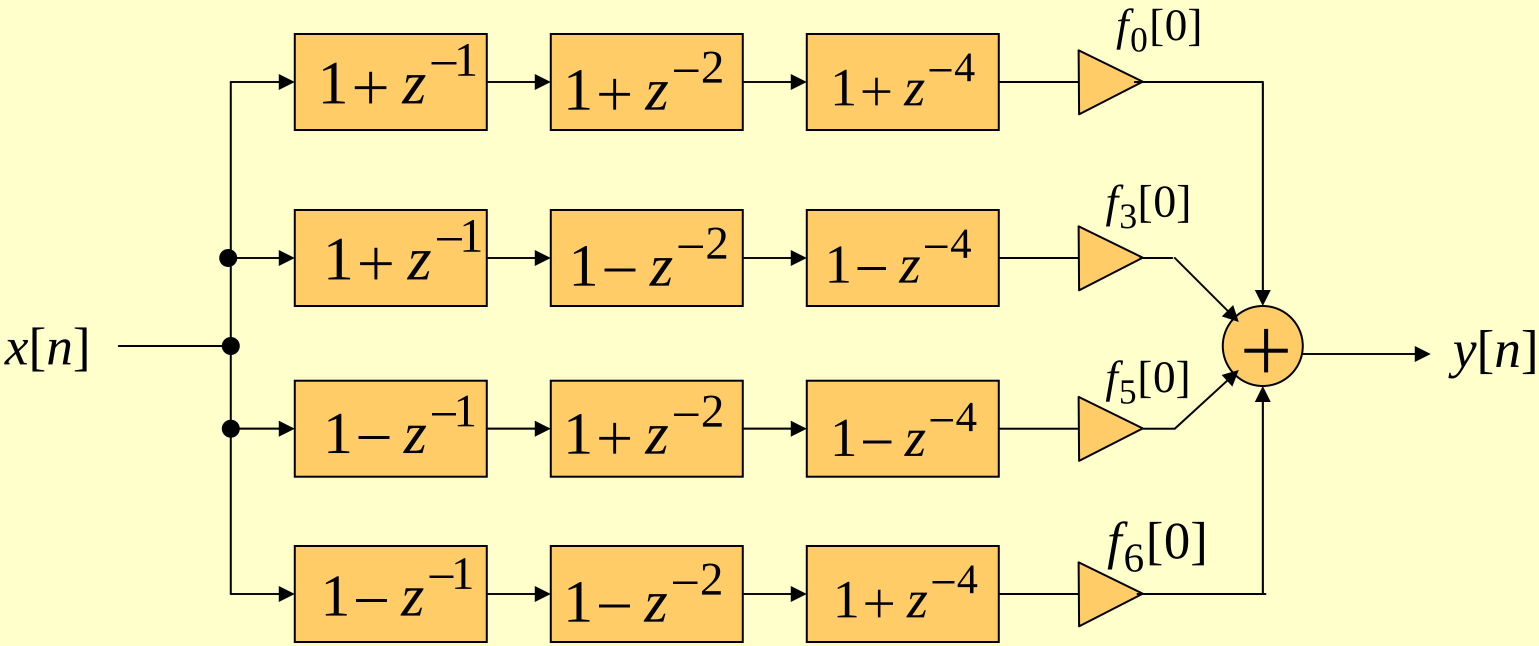
- For $L = M$, the coefficients $f_k[n]$ of the subfilters $F_k(z)$ can be expressed in terms of the coefficients $\{h[n]\}$ of the overall filter $H(z)$:

$$\begin{bmatrix} f_k[0] \\ f_k[1] \\ \vdots \\ f_k[P-1] \end{bmatrix} = \frac{1}{M} \cdot \mathbf{R}_M \cdot \begin{bmatrix} h[k] \\ h[k+M] \\ \vdots \\ h[k+M(P-1)] \end{bmatrix}$$

- Each subfilter has, in general, P non-zero coefficients

Efficient FIR Filter Implementation

- Simpler realizations are obtained in the case of linear-phase FIR filters
- The 4-branch realization of a length-8 type 2 FIR filter is shown below



Efficient FIR Filter Design

- The structural subband decomposition of an FIR transfer function $H(z)$ simplifies considerably the filter design process

- To this end, two different design approaches have been advanced

Efficient FIR Filter Design

- In one approach, each branch is designed one-at-a-time using either a least-squares minimization method or a minimax optimization method

- In the other approach, each subfilter is designed using a frequency sampling method

Efficient FIR Filter Design

- Let $\mathcal{H}(\omega)$ denote the amplitude function of a linear-phase frequency response
- For the parallel IFIR structure we then have

$$\mathcal{H}(\omega) = \sum_{k=0}^{M-1} \mathcal{G}_k(\omega)\mathcal{F}_k(\omega M)$$

where $\mathcal{G}_k(\omega)$ and $\mathcal{F}_k(\omega M)$ are the amplitude functions of the k-th interpolator and the k-th sub-filter, respectively

Efficient FIR Filter Design

- Filter design problem - Determine the $N/2M$ coefficients of each sparse subfilter $F_k(z^M)$ for $k = 0, 1, \ldots, M-1$ to approximate a specified $\mathcal{H}(\omega)$

Efficient FIR Filter Design

Least-squares optimization -

- By taking the samples of the respective amplitude functions at D suitably chosen discrete frequency points in the interval $0 \le \omega \le \pi$, we can write

$$\tilde{\mathbf{h}} = \sum_{k=0}^{M-1} \tilde{\mathbf{G}}_k \tilde{\mathbf{f}}_k$$

Efficient FIR Filter Design

- where

  $\tilde{\mathbf{h}}$ - a vector representing the discretized version of $\mathcal{H}(\omega)$

  $\tilde{\mathbf{G}}_k$ - a diagonal matrix with diagonal elements given by samples of $\mathcal{G}_k(\omega)$

  $\tilde{\mathbf{f}}_k$ - a column vector containing samples of $\mathcal{F}_k(\omega M)$

Efficient FIR Filter Design

- If $\tilde{\mathbf{h}}_d$ denotes the desired amplitude response samples of the parallel IFIR structure, the approximation error is then given by

$$\mathbf{e} = \tilde{\mathbf{h}}_d - \tilde{\mathbf{h}} = \tilde{\mathbf{h}}_d - \sum_{k=0}^{M-1} \tilde{\mathbf{G}}_k \tilde{\mathbf{f}}_k$$

- Design objective - Minimize the $\mathcal{L}_2$-norm of **e** separately with respect to each of the sub-filters

Efficient FIR Filter Design

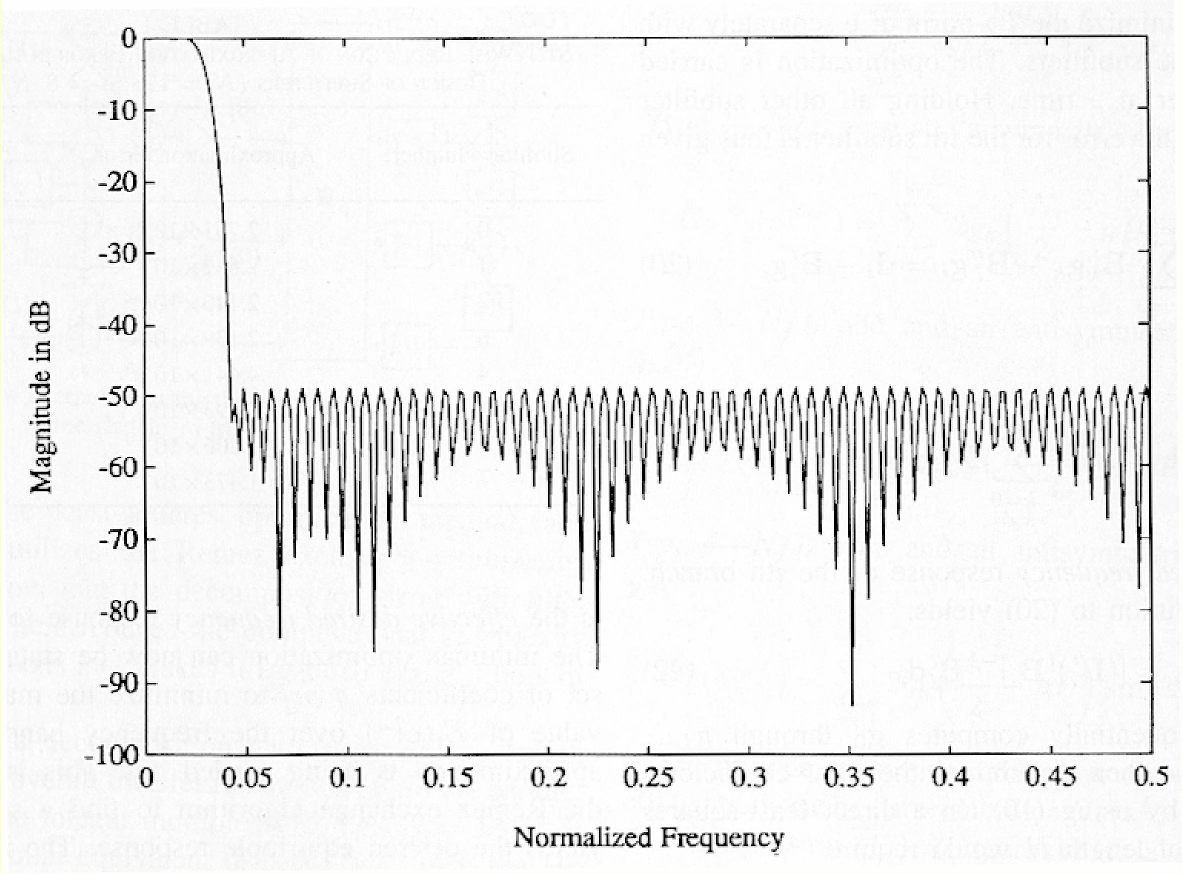
- The minimization procedure results in the determination of the coefficients $f_k[n]$ of all sub-filters from which the impulse response samples of the overall filter can be obtained
- The computational complexity of the modified least-squares method is smaller by a factor of $1/M$ compared to that of the direct least-squares method

Efficient FIR Filter Design

- Example - Design a linear-phase lowpass FIR filter of length 128 using an 8-band decomposition
- Filter specifications: passband edge at 0.02π and stopband edge at 0.04π
- The gain response of the filter designed using the least-squares approach is shown on the next slide

Efficient FIR Filter Design

- Gain response



Efficient FIR Filter Design

Minimax optimization -

- Here, the weighted error of approximation for a linear-phase filter design is given by

$$E(\omega) = W(\omega)[H_d(\omega) - \sum_{k=0}^{M-1} G_k(\omega) F_k(\omega M)]$$

where $H_d(\omega)$ is the desired amplitude response and $W(\omega)$ is a weighting function

Efficient FIR Filter Design

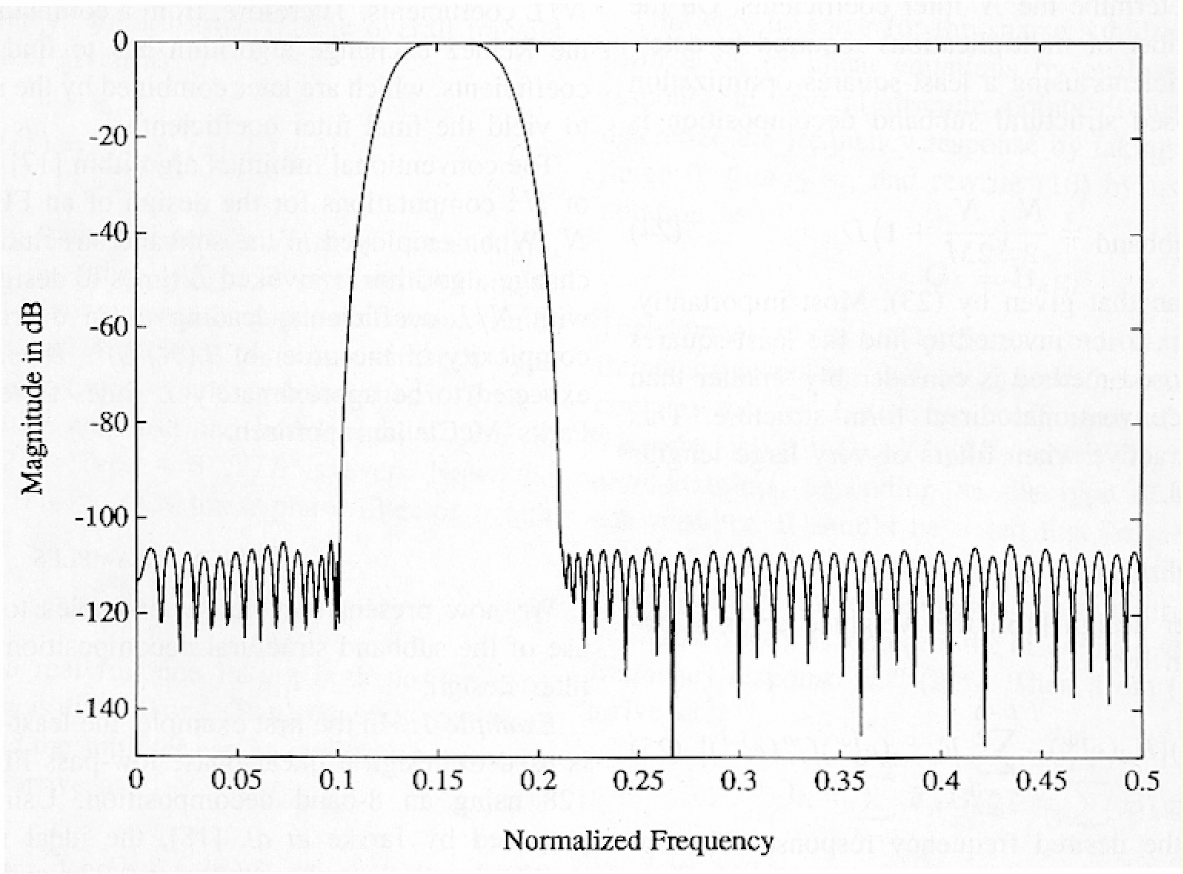
- The optimization is carried out over one subfilter at a time using the Remez method
- The computational complexity of the structural subband based method is smaller by a factor of $1/M$ compared to that of the Parks-McClellan method

Efficient FIR Filter Design

- Example - Design a bandpass FIR filter of length with passband edges at 0.15π and 0.16π, and stopband edges at 0.1π and 0.21π, respectively
- Passband and stopband ripples are assumed to have equal weights
- Assume a 8-band decomposition

Efficient FIR Filter Design

- Gain response



Efficient FIR Filter Design

- It is possible to design a nearly optimum FIR filter, based on a 2-band Hadamard-matrix based structural subband decomposition, by applying the minimax routine to each of the two smaller size subfilters without repeated iterations and combining the paths

Efficient FIR Filter Design

Frequency-sampling approach

- Here, simple analytical expressions for the passband, transition band, and the stopband are first sampled at equally-spaced points on the unit circle to arrive at the original frequency samples, $\hat{H}(m),\ 0 \le m \le N-1$, of the overall parallel IFIR structure

Efficient FIR Filter Design

- From $\hat{H}(m)$ the desired frequency samples of the subfilters, $\hat{F}_k(\ell)$, $0 \le \ell \le P-1$, $0 \le k \le M-1$, are then determined using

$$\begin{bmatrix} \hat{F}_0(\ell) \\ \hat{F}_1(\ell) \\ \vdots \\ \hat{F}_{M-1}(\ell) \end{bmatrix} = \mathbf{T}^{-1} \cdot \mathbf{B}^{-1} \cdot \mathbf{W}_M^{-1} \cdot \begin{bmatrix} \hat{H}(\ell) \\ \hat{H}(\ell+P) \\ \vdots \\ \hat{H}(\ell+P(M-1)) \end{bmatrix}$$

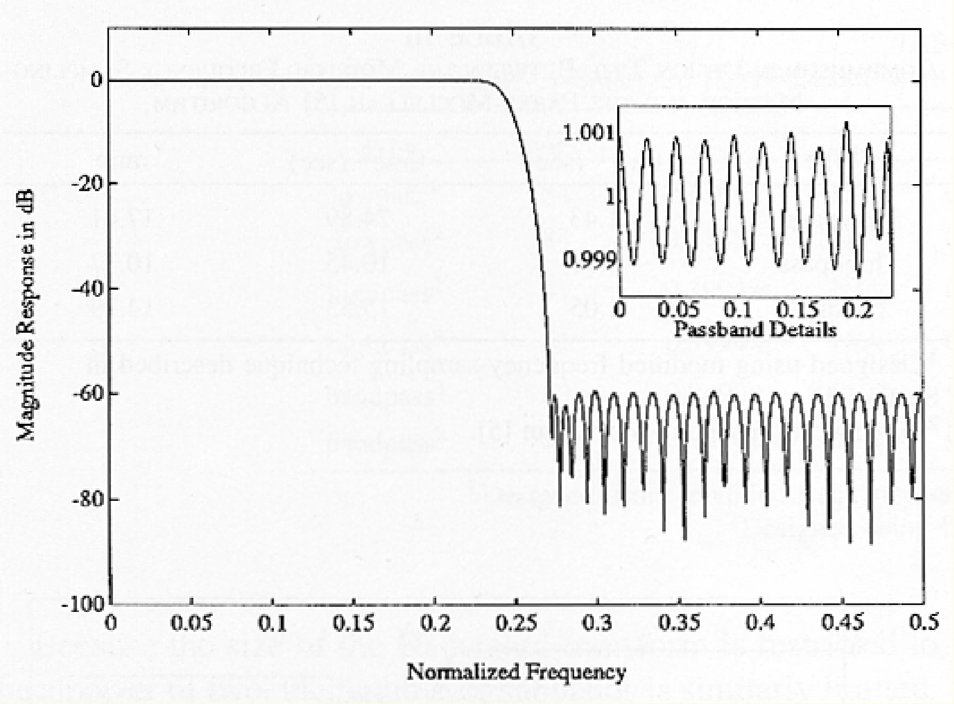
where $\mathbf{B} = \mathrm{diag}[1 \quad W_N^{\ell} \quad \cdots \quad W_N^{(M-1)\ell}]$
and $\mathbf{W}_M$ is an $M \times M$ DFT matrix

Efficient FIR Filter Design

- An IDFT of the vector of the frequency samples of each subfilter yields its impulse response samples
- Example - Design a half-band FIR filter with a passband ripple of $\delta_p = 0.0013$ and a stopband ripple of $\delta_s = 0.001$ using a 4-band decomposition

Efficient FIR Filter Design

- Gain response

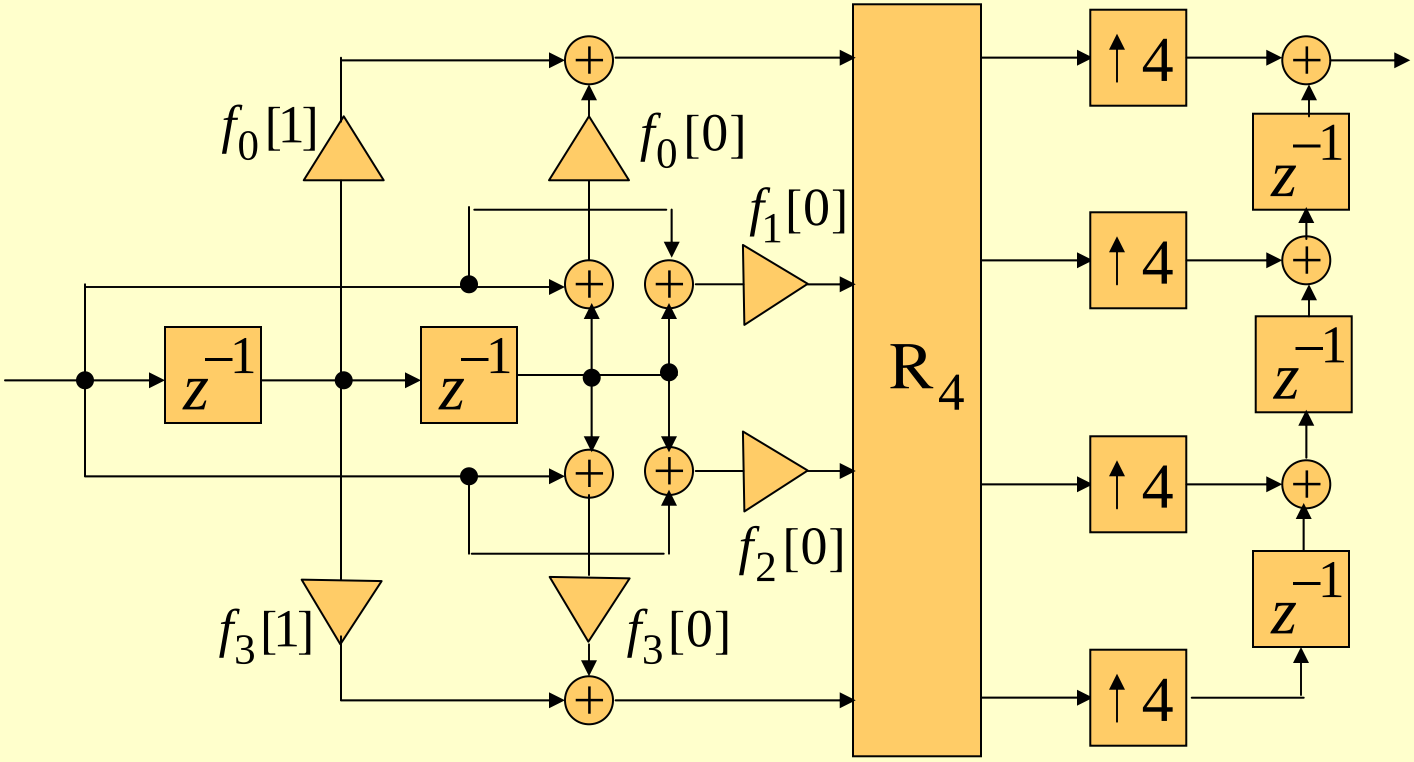


Efficient Decimator and Interpolator Structures

- Structural sub-band decomposition-based structure can be computationally more efficient than the conventional polyphase decomposition-based structure in realizing decimators and interpolators employing linear-phase Nyquist filters
- To this end, it is necessary to use transform matrices that transfer the filter coefficient symmetry to the sub-filters

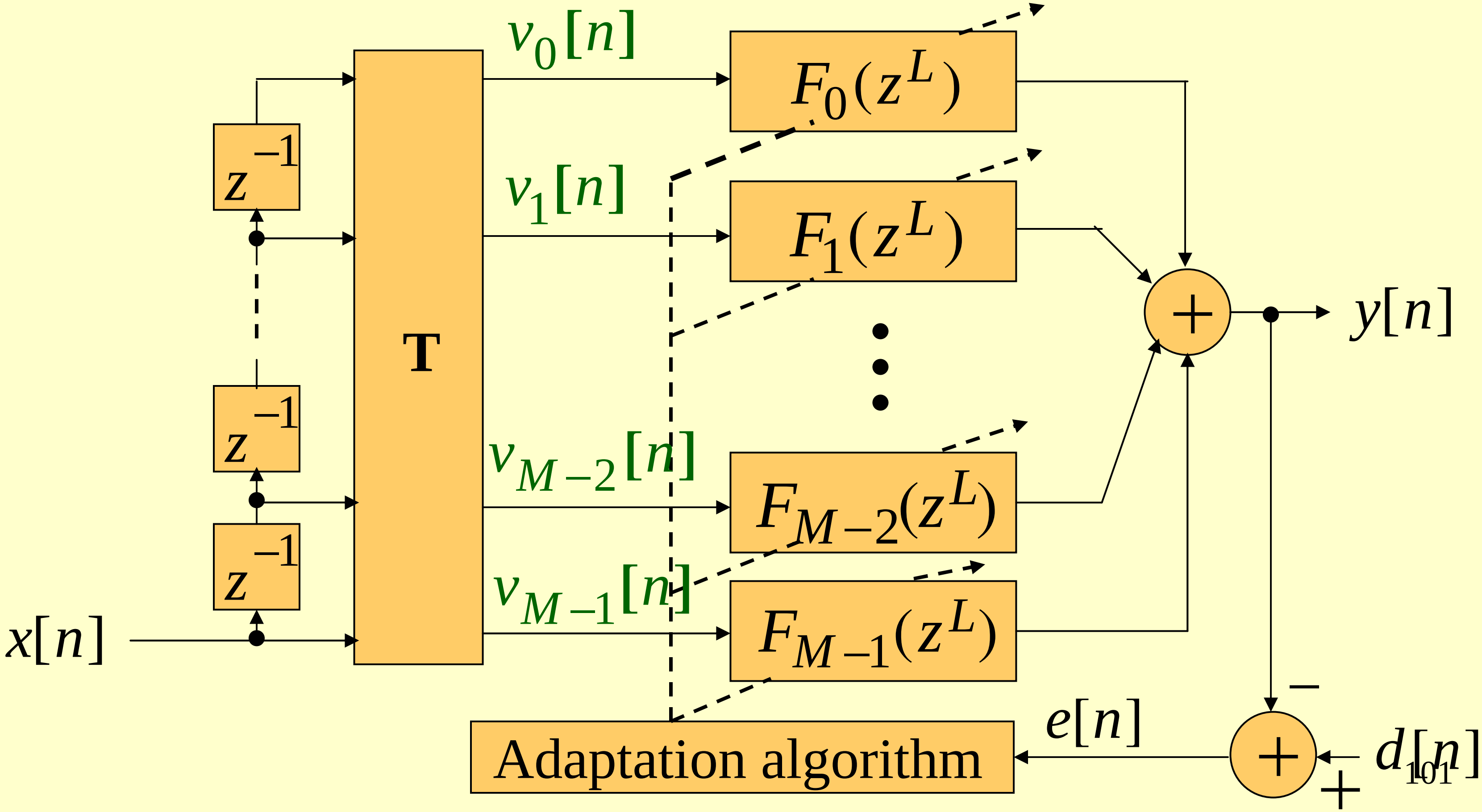
Efficient Decimator and Interpolator Structures

- A factor-of-4 interpolator structure



Subband Adaptive Filtering

- Based on the generalized structural sub-band realization



Subband Adaptive Filtering

- Here, the input signal $x[n]$ is first processed by a fixed $M \times M$ unitary transform $\mathbf{T}$, generating the signals $v_i[n]$, which are then filtered by the sparse adaptive sub-filters $F_i(z^L)$

Subband Adaptive Filtering

- For large values of M, recursive DFT or DCT algorithms are computationally more efficient to implement the transform $\mathbf{T}$ than the FFT-type algorithms
- For small values of M, dedicated fast non-recursive algorithms are preferred to implement the transform $\mathbf{T}$

Subband Adaptive Filtering

- The output $y[n]$ can be expressed as

$$y[n] = \sum_{\ell=0}^{M-1} \mathbf{v}^T[n-\ell L]\cdot \mathbf{f}_\ell[n]$$

where

$$\mathbf{v}[\mathrm{n}] = \left[v_0[n] \quad v_1[n] \quad \cdots \quad v_{M-1}[n]\right]^T$$

is the vector of transformed inputs, and

$$\mathbf{f}_\ell[n] = \left[f_{0,\ell}[n] \quad f_{1,\ell}[n] \quad \cdots \quad f_{M-1,\ell}[n]\right]^T$$

is the subfilter coefficient vector containing the ℓ-th coefficient of each sub-filter

Subband Adaptive Filtering

Normalized LMS Algorithm -

- The subfilter coefficient vector update equation is given by

$$f_\ell[n+1] = f_\ell[n] + 2\mu\Lambda^2 e[n]\mathrm{v}^*[n-\ell L],$$
$$\ell = 0,1,\ldots,P-1$$

- where μ is the adaptation step size, and Λ^2 is an $M \times M$ diagonal matrix containing the power estimates of $v_i[n]$

Subband Adaptive Filtering

- For $M = L = N$, i.e., P =1 (in which case each of the sub-filters consists of a single coefficient), the proposed method reduces to the transform-domain LMS algorithm

- For $M = L = 1$, and $\mathbf{T} = 1$, the proposed method reduces to the conventional time-domain LMS algorithm

Subband Adaptive Filtering

- The sub-band adaptive filter structure offers additional flexibility in the choice of the number of sub-bands M and the sparsity factor L
- This feature is attractive in the case of higher-order adaptive filters, as it provides a reduction in the computational complexity compared to the transform-domain algorithm and improved convergence performance compared to the LMS algorithm

Subband Adaptive Filtering

- Choice of a transform T with good frequency selection decreases the correlation among the transformed signals, which can be used to obtain a significant improvement in the convergence speed of the LMS algorithm for colored input signals
- In these cases, the DFT or DCT have been found to be useful

Subband Adaptive Filtering

- The contribution of each sub-filter is mainly restricted to a frequency sub-band, which can be used advantageously to increase the speed of convergence of the adaptive algorithm
- The structure also has the flexibility of allowing sub-bands not contributing greatly to the overall frequency response to be removed, reducing the number of operations needed for the filter implementation

Subband Adaptive Filtering

- Example - We examine the behavior of the subband adaptive line enhancer (ALE)
- Input consists of a single sinusoid of unit amplitude plus white Gaussian noise with a variance 0.25 (SNR = 3 dB)
- We choose $N = 128$, $M = 8$, $P = 16$
- For a DCT transform matrix we choose $L = 8$
- For a DFT transform matrix we choose $L = 4$

Subband Adaptive Filtering

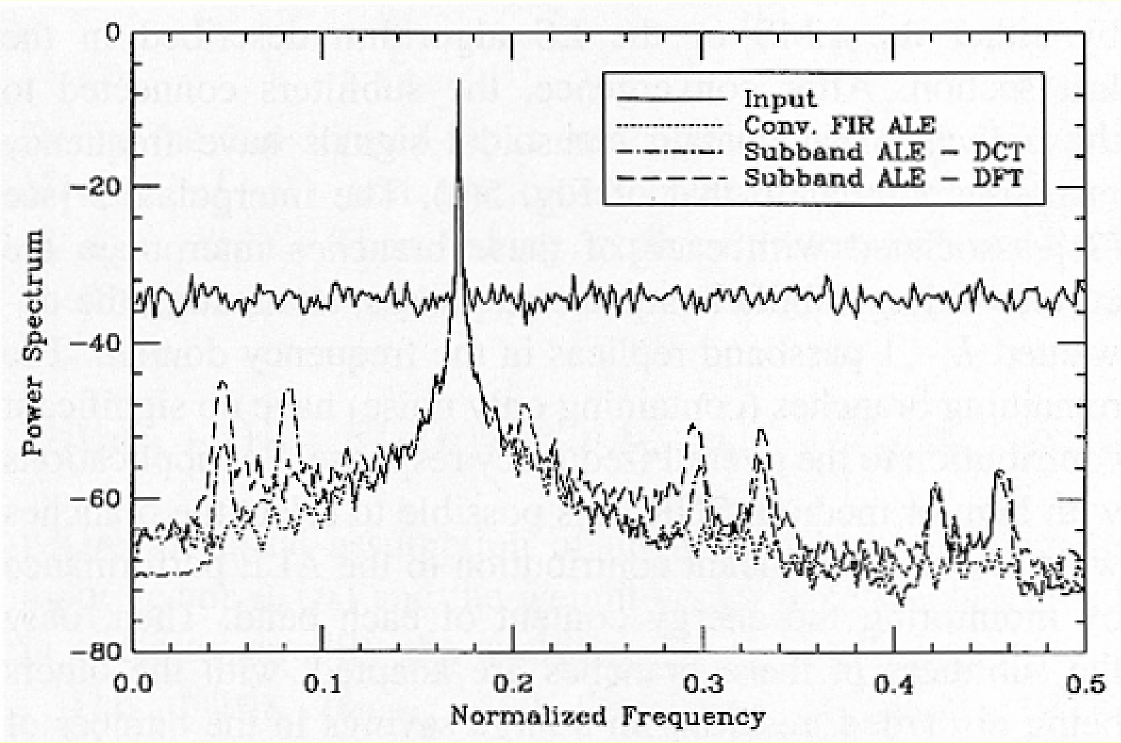
- The coefficients were updated using the LMS algorithm
- The output power spectra estimated using averaged periodograms of 16 data blocks of length 512 for the different ALE structures
- In the DCT structure, 2 bands out of 8 were kept
- In the DFT structure, 1 band out of 8 were kept

Subband Adaptive Filtering

- In both DCT and DFT cases, the number of operations required for the ALE implementation was about 1/4-th of those required in the conventional ALE implementation
- Further savings in the number of operations in the subband ALE approach results when a frequency estimate of the input sinusoid is required

Subband Adaptive Filtering

- Output power spectra for $\omega_o = 0.17$



Subband Adaptive Filtering

- Output power spectra for the subband ALE structures show some minor peaks due to band removals which may be acceptable in most applications
- Subband ALE approch has been used in acoustic echo cancellation and adaptive channel equalization